"Palatini's Cousin: A New Variational Principle"

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Abstract:
A variational principle is suggested within Riemannian geometry, in which an auxiliary metric and the Levi Civita connection are varied independently. The auxiliary metric plays the role of a Lagrange multiplier and introduces non-minimal coupling of matter to the curvature scalar. The field equations are 2nd order PDEs and easier to handle than those following from the so-called Palatini method. Moreover, in contrast to the latter method, no gradients of the matter variables appear. In cosmological modeling, the physics resulting from the new variational principle will differ from the modeling using the Palatini method.

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1 Introduction

For the derivation of the field equation of Einstein's theory of gravitation and of alternative gravitational theories sometimes a method named, alternatively, "Palatini's Principle", "the Palatini method of variation" or "Palatini's device" is used. Although the starting point is Riemannian geometry, besides the metric an independent affine connection forming the curvature tensor is imagined; in the Lagrangian, both metric and connection then are varied independently. An advantage of the method is that it leads to 2nd order field equation for Lagranians of higher order in curvature while a variation of the metric as the only variable results in 4th-order PDEs. On the other hand, a main conceptual difficulty of the method is that the variational procedure mixes Riemannian and metric-affine geometry. Authors either leave undetermined the space-time geometry as a frame for the new connection, or tacitely fix it mentally by introducing constraints (symmetric connection, no torsion etc) which do not show up in the formalism.

Since many years, warnings have been voiced that the method be working reliably only for the Hilbert-Einstein Lagrangian (plus the matter part) $\mathcal{L} = \sqrt{-g}\ [R(g_{ij}) + 2\kappa L_{mat}(g_{ij}, u^A)]$ with curvature scalar $R = g^{lm}R_{lm}(g_{ij})$ and matter variables u^A, but otherwise leads to under- and un-determinacies [1], [2], [3].[1] Recently, Palatini's method has been unearthed in attempts to build cosmological models thought to explain the accelerated expansion of the universe with its consequences for dark energy [7], [8], [9], [10], [11]. The method also has been applied to loop quantum cosmology [12]. Often, the starting point is a Lagrangian of the form $\mathcal{L} = \sqrt{-g}\ [\ R(g_{ij}) + \tilde{f}(R)\ ] + \sqrt{-g}\ 2\kappa L_{mat}(g_{ij}, u^A)$ with $\tilde{f}$ an arbitrary smooth function.[2] In the following, we suggest another variational principle leading to 2nd order field equations and lacking the deficiencies of the Palatini method. After its introduction, it is applied to the class of $f(R)$-theories in section 3 and compared with the Palatini method in section 4. A recent particular choice for $f(R)$ in the framework of cosmological modeling then is used as an example for the working of the new principle.

2 The new variational principle

Whereas in the Palatini method the Levi Civita connection (represented by the Christoffel symbol) is replaced by a general affine connection, here we keep the geometry (pseudo-)Riemannian but introduce an auxiliary Lorentz metric. This is done by replacing, in an action integral set up within Riemannian geometry, the (Lorentz-)metric g_{ab} by an auxiliary metric γ_{ab} except in the Levi Civita connection which is left unchanged. The independent variables for the variation are γ_{ab} and the Levi Civita connection formed from g_{ab}

$$\{^{\ k}_{ij}\}_g = \frac{1}{2} g^{kl} (\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l})\ . \tag{1}$$

The equations following from the variation will give the dynamics of the gravitational field and link γ_{ab} with g_{ab}. We wish to emphasize that it is *not*

[1] For incorrectly relating Palatini's name with what is ascibed to him cf. [4], footnote on p. 40 as well as the English translation of Palatini's paper in the same volume on pp. 477-488 (1980). Cf. also [5].

[2] Recently, Lagrangians with two curvature invariants, i.e., $f(R,\ R_{ab}R_{lm}g^{al}g^{bm})$ have been considered [13].

a bi-metric theory which is aimed at.[3] The auxiliary metric may be seen as playing the role of a Lagrange multiplier. This is analogous to the case of scalar-tensor theories replacing $f(R)$-theories of gravitation (cf. [14], [15]). The new variational principle for Einstein gravity starts from:[4]

$$\mathcal{L} = \sqrt{-\gamma}\ [\gamma^{ab}\ R_{ab}(\{^{k}_{ij}\}_g) + 2\kappa L_{mat}(\gamma_{lm}, u^A)]\ . \tag{2}$$

Variation with respect to γ^{ab} leads to:

$$\delta_\gamma \mathcal{L} = \sqrt{-\gamma}\ [\ R_{ab}(\{^{k}_{ij}\}_g) - \frac{1}{2}\gamma_{ab}\ R_\gamma + 2\kappa\ T_{ab}(\gamma_{lm}\ , u^A)\ ]\ \delta\gamma^{ab}, \tag{3}$$

where $R_\gamma := \gamma^{lm} R_{lm}(\{^{k}_{ij}\}_g)$ and $T_{ab} := \frac{2}{\sqrt{-\gamma}}\ \frac{\delta \mathcal{L}_{mat}}{\delta\gamma^{ab}}$. Variation with respect to $\{^{k}_{ij}\}_g$ gives:

$$\delta_{\{^{k}_{ij}\}_g} \mathcal{L} = [\ -(\sqrt{-\gamma}\ \gamma^{b(i})_{;b}\delta_k^{\ j)} + (\sqrt{-\gamma}\ \gamma^{ij})_{;k}\ ]\ \delta(\{^{k}_{ij}\}_g) \tag{4}$$

up to divergence terms.[5] From $\delta_{\{^{k}_{ij}\}_g}\mathcal{L} = 0$, after a brief calculation using the trace of (4),

$$(\sqrt{-\gamma}\ \gamma^{ij})_{;k} = 0 \tag{5}$$

follows, where the covariant derivative is formed with the Levi Civita connection. Thus, $\gamma^{ab} = const \cdot g^{ab}$ follows. $\delta_\gamma \mathcal{L} = 0$ from (3) reduces to Einstein's field equations.

The method is particularly well suited to a calculus with differential forms. Here, the usual basic 1-forms $\theta^i = e^i_{\ r}\ dx^r$ and the curvature 2-form $\Omega_{ij} = \frac{1}{2} R_{ijkl}(g_{lm})\theta^k \wedge \theta^l$ are taken as the independent variables. In place of the auxiliary metric γ_{ij}, now an auxiliary 1-form is introduced and denoted by $\bar{\theta}^i = \bar{e}^i_{\ r}\ dx^r$ where

$$\bar{e}^i_{\ r}\ \bar{e}^j_{\ s}\ \eta^{ij} = \gamma^{rs},\ e^i_{\ r}\ e^j_{\ s}\ \eta^{ij} = g^{rs}\ . \tag{6}$$

[3] In bi-metric theories, one metric usually is fixed to be the flat Minkowskian metric and not varied. A formal variation of the second metric often is restricted to an infinitesimal coordinate change in order to derive conservation laws. Cf. [6].

[4] Latin indices a, b, i, j, ... run from 0 to 3; the summation convention is implied.

[5] $(\sqrt{-\gamma}A^k)_{;k}$ always may be written as $\sqrt{-g}\ (\sqrt{\frac{\gamma}{g}}A^k)_{;k}$ and thus as $(\sqrt{-\gamma}A^k)_{,k}$.

The Einstein-Hilbert Lagrangian is $\mathcal{L}_E = \Omega_{ab} \wedge *(\bar{\theta}^a \wedge \bar{\theta}^b)$ with the Hodge-star operation: $*(\bar{\theta}^a \wedge \bar{\theta}^b) =: \bar{\epsilon}^{ab}$ and $\bar{\epsilon}_{ab} := \frac{1}{2!}\epsilon_{ablm}\bar{\theta}^l \wedge \bar{\theta}^m$.[6] Variation with regard to the fundamental 1-forms and curvature form leads to the field equations:

$$D(\frac{\partial \mathcal{L}_E}{\partial \Omega_{ij}}) = 0, \quad \frac{\partial \mathcal{L}_E}{\partial \bar{\theta}_i} = 0 \tag{7}$$

with the covariant external derivative D using the Levi Civita connection (1-form). Because of $\frac{\partial \mathcal{L}_E}{\partial \Omega_{ij}} = *(\bar{\theta}^i \wedge \bar{\theta}^j)$ and of $\frac{\partial \mathcal{L}_E}{\partial \bar{\theta}_i} = \Omega_{lm} \wedge \bar{\epsilon}_{ilm}$, the field equations are:

$$D\bar{\epsilon}^{ij} = 0 \ , \ \Omega_{lm} \wedge \bar{\epsilon}_{ilm} = 0 \ , \tag{8}$$

where $\bar{\epsilon}_{ilm} := \epsilon_{ilmp}\theta_p$ is a 1-form; $\bar{\epsilon}^{ilm}$ is dual to $\bar{\theta}^i \wedge \bar{\theta}^l \wedge \bar{\theta}^m$. Standard manipulations with the forms show that the 1st equation (8) is satisfied identically due to the absence of torsion, i.e, $D\bar{\theta}^m = 0$; and that the 2nd becomes: $2G^c{}_a(g)\bar{\epsilon}_c = 0$ with the Einstein tensor $G^c{}_a(g)$ and the 3-form $\bar{\epsilon}_i := \frac{1}{3!}\epsilon_{iklm}\bar{\theta}^k \wedge \bar{\theta}^l \wedge \bar{\theta}^m$. An advantage of this formalism is that it may be adapted easily to gauge theories.

3 Extension to f(R)-theories

The new variational principle easily applies to the Lagrangian

$$\mathcal{L} = \sqrt{-\gamma}\ [f(\gamma^{lm}R_{lm}(\{{}^{k}_{ij}\}_g)) + 2\kappa\ \mathcal{L}_{mat}(\gamma_{ij}, u^A)]\ . \tag{9}$$

The variations lead to:

$$\delta_\gamma \mathcal{L} = \sqrt{-\gamma}\ [\ f'(R_\gamma)R_{ab}(\{{}^{k}_{ij}\}_g) - \frac{1}{2}\gamma_{ab}\ f(R_\gamma) + 2\kappa\ T_{ab}(\gamma_{lm}\ , u^A, \partial u^A)\ ]\ \delta\gamma^{ab}, \tag{10}$$

whith $f' := \frac{df}{dR}$ and to

$$\delta_{\{{}^{k}_{ij}\}_g}\mathcal{L} = [\ -(\sqrt{-\gamma}\ f'(R_\gamma)\gamma^{b(i})_{;b}\delta_k^{j)} + (\sqrt{-\gamma}\ f'(R_\gamma)\gamma^{ij})_{;k}\ ]\ \delta(\{{}^{k}_{ij}\}_g) \tag{11}$$

up to divergence terms. As in section 2, from (11)

$$(\sqrt{-\gamma}\ f'(R_\gamma)\gamma^{ij})_{;k} = 0\ , \tag{12}$$

[6] Notation here is somewhat ambiguous: e.g., the curvature form depends on both the Levi Civita connection and the auxiliary tetrad: $\Omega_{ij} = \frac{1}{2}R_{ijkl}(\{{}^{k}_{ij}\}_g)\ \bar{\theta}^k \wedge \bar{\theta}^l$. Nevertheless, no bar will be put on Ω. The notation $\Omega_{ij}(g, \bar{\theta})$ would be inconvenient.

but from which now follows:

$$\gamma^{ab} = f'(R_\gamma) g^{ab}, \quad \gamma_{ab} = (f'(R_\gamma))^{-1} g_{ab} \ . \tag{13}$$

From (13), $R_\gamma = f'(R_\gamma) R_g$ with $R_g := g^{lm} R_{lm}(\{^{k}_{ij}\}_g)$, i.e., the curvature scalar in (pseudo-)Riemannian space-time. Writing

$$R_g = \frac{R_\gamma}{f'(R_\gamma)} =: r(R_\gamma) \ , \tag{14}$$

the relation $R_\gamma = r^{-1}(R_g)$ can be used to remove all entries of γ_{ab} via the curvature scalar in the field equations following from (10). Expressed by g_{ab}, they read as:

$$f'(r^{-1}(R_g)) \, R_{ab}(\{^{k}_{ij}\}_g) - \frac{1}{2} \, g_{ab} \, \frac{f(r^{-1}(R_g))}{f'(r^{-1}(R_g))} + 2\kappa \, T_{ab}((f')^{-1}(r^{-1}(R_g)) \, g_{lm} \, , u^A) = 0 \ . \tag{15}$$

Equation (15) shows that, in contrast to f(R)-theories leading to 4th-order differential equations when derived by variation of only the metric g_{ab}, the new field equations are of 2nd order in the derivatives of g_{ab}. The auxiliary metric is fully determined: $\gamma^{ab} = f'(r^{-1}(R_g)) g^{ab}$; it is not an absolute object. Beyond acting as a Lagrange multiplier its main function is its appearance in the matter tensor causing non-minimal coupling to the curvature scalar. No further role in the description of the gravitational field is played.[7] For a Lagrangian of the form $\sqrt{-g} \ [\ R(g_{ij}) + \tilde{f}(R) \ ]$, in the formalism given above f is to be replaced by $R + \tilde{f}(R)$, f' by $1 + \tilde{f}'$ while $f'' = \tilde{f}''$, $f''' = \tilde{f}'''$.

A.

First, a non-vanishing trace (with respect to the auxiliary metric γ) of the matter tensor will be assumed $T_\gamma := \gamma^{lm} T_{lm}(\gamma_{rs} \ , u^A) \neq 0$. In this case, the curvature scalar is seen to be a functional of the trace of the matter tensor. Because of

$$T_\gamma = f'(R_\gamma) \ g^{lm} T_{lm}(f'(r^{-1}(R_g)) g_{rs}, u^A) = f'(r^{-1}(R_g)) T_g(f'(r^{-1}(R_g)) g_{rs}, u^A), \tag{16}$$

with $\tilde{T}_g := g^{lm} T_{lm}(\gamma_{rs}, u^A)$ from the g-trace of (15) follows:

$$f'^2 \ R_g - 2f + 2\kappa \ f' \ \tilde{T}_g = 0 \ , \tag{17}$$

[7] In particular, γ^{ab} does *not* enter the Levi Civita connection, but only the matter tensor. As a metric γ_{ab} is incompatible with the Levi Civita connection; its non-metricity tensor does not vanish.

or, precisely,

$$(f'(r^{-1}(R_g))^2 R_g - 2f(r^{-1}(R_g)) + 2\kappa f'(r^{-1}(R_g))\tilde{T}_g((f'(R_g))^{-1} g_{lm} u^A) = 0 \ . \tag{18}$$

With a newly defined function ω this can be written as

$$R_g = \omega(2\kappa T_g) \ , \tag{19}$$

where now $T_g := g^{lm} T_{lm}(g_{rs}, u^A)$. From (18) we conclude that (15) can be cast into the form of Einstein's equations with an effective matter tensor. The curvature scalar is coupled directly to the matter variables showing up in its trace; no derivatives are involved. In fact:

$$R_{ab}(\{{}^{k}_{ij}\}_g) - \frac{1}{2}\, g_{ab}(R_g) = -\frac{2\kappa}{f'}\, [\, T_{ab}(\frac{1}{f'}\, g_{lm}, u^A) - \frac{1}{2} T g_{ab}\,] - \frac{1}{2} g_{ab}\, \frac{f}{(f')^2} \ . \tag{20}$$

In the case of *perfect fluid* matter with energy density μ and pressure p

$$T_{ab}(\gamma_{rs}, u^A) = (\mu + p)\ \gamma_{al}\gamma_{bm}\bar{u}^l \bar{u}^m - p\ \gamma_{ab} \tag{21}$$

with $\bar{u}^l := \frac{dx^l}{d\bar{s}}$ and $d\bar{s}^2 = \gamma_{lm} dx^l dx^m$. Hence, $\bar{u}^l = (f')^{1/2}\ u^l\ , u^l = \frac{dx^l}{ds}$ and

$$T_{ab}(\gamma_{rs}, u^A) = (f')^{-1}(R_g)\ T_{lm}(g_{rs}, u^A) \ . \tag{22}$$

In this case, from (16) a simple relationship for the γ- and g-traces of the matter tensor follows:

$$T_\gamma(\gamma_{rs}\ , u^A) = T_g(g_{rs}, u^A) = \mu - 3p \ . \tag{23}$$

In place of $T^{ab}{}_{;\,b} = 0$ for the Einstein-Hilbert Lagrangian, in this theory a more general relationship with $T^{ab}{}_{;\,b} \neq 0$ follows from general covariance. This is also seen by forming the divergence of the Einstein tensor in (20).

B.
For vanishing trace of the matter tensor $T_\gamma = 0$, (18) reduces to

$$f'(R_\gamma) R_\gamma - 2f(R_\gamma) = 0 \ . \tag{24}$$

This implies two cases:
i) $f = (f_0 R_\gamma)^2$, and ii) $f \neq (f_0 R_\gamma)^2$. The exceptional case i) is characterized

by an additional scale invariance implying zero trace for the matter tensor. The field equations (15) become

$$2(f_0)^2(R_\gamma)[R_{ab}(g) - \frac{1}{4}R_g\ g_{ab}] + 2\kappa T_{ab}(\frac{1}{2f_0^2 R_\gamma}\ g_{lm}, u^A) = 0\ ,$$
$$R_g = \frac{1}{2(f_0)^2}\ ,\ T_\gamma = T_g = 0\ . \tag{25}$$

If we take a sourceless Maxwell field as matter, then

$$T_{ab}(\gamma_{lm}, F_{lm}) = \gamma^{lm}F_{al}F_{bm} - \frac{1}{4}\gamma_{ab}\gamma^{il}\gamma^{jm}F_{il}Fjm = f'(R_\gamma)T_{ab}(g_{lm}, F_{lm})\ . \tag{26}$$

R_γ drops out and the field equations are:

$$R_{ab}(g) - \frac{1}{4}R_g\ g_{ab} + \kappa\ T_{ab}(g_{lm}, u^A) = 0\ ,$$
$$R_g = \frac{c_1}{f'(c_1)}\ ,\ T_\gamma = T_g = 0\ . \tag{27}$$

In case ii), from (24) $R_\gamma = c_1 = const$ and we may proceed only if one real solution of $f'(c_1)\ c_1 - 2f(c_1) = 0$ does exist and if $f(c_1),\ f'(c_1)$ remain finite. The field equations then are

$$f'(c_1)R_{ab}(g) - \frac{c_1}{4}\ g_{ab} + 2\kappa T_{ab}(\frac{1}{f'(c_1)}\ g_{lm}, u^A) = 0\ ,$$
$$T_\gamma = T_g = 0\ . \tag{28}$$

In Einstein's theory, $R = 0$ follows if the trace of the matter tensor is vanishing. Here, the larger set of solutions $R = const$ is obtained.

Above, it has been assumed that the matter tensor does not contain covariant derivatives; this covers most cases of physical interest. Otherwise, formidable complications result even when the Einstein-Hilbert Lagrangian is taken. E.g., if the additional term in the matter Lagrangian is $\sim \sqrt{-\gamma}\gamma^{il}\gamma^{km}u_{i;k}\ u_{l;m}$ (5) must be replaced by $(\sqrt{-\gamma}\ \gamma^{ij})_{;k} = f_k^{ij}(\gamma^{lm}, u^A, \partial u^A)$ with a particular functional f_k^{ij}. Hence, the elimination of the Lagrangian multiplier will requirec quite an effort.

4 Comparison with the Palatini method

For the Palatini method of variation with variables g_{ij} and Γ^k_{ij}, the field equations of the f(R)-theory are:

$$f'(R)R_{ik}(\Gamma) - \frac{1}{2}f(R)g_{ik} = -2\kappa\ T_{ik}, \tag{29}$$

$$(\sqrt{-g}f'(R)g^{il})_{\|l} = 0\ , \tag{30}$$

where the covariant derivative is formed with the connection Γ and $R = g^{ik}R_{ik}(\Gamma)$. From (30) we obtain a metric $\bar{g}_{ij}$ compatible with the connection Γ:

$$\bar{g}_{ij} = f'(R)\ g_{ij}\ ,\ \bar{g}^{ij} = (f')^{-1}(R)\ g^{ij}\ , \tag{31}$$

and the relation between Γ and the Levi Civita connection is:

$$\Gamma^k_{ij} \equiv \{^{\,k}_{ij}\}_{\bar{g}} = \{^{\,k}_{ij}\}_g + \frac{1}{2}\frac{d}{dR}(lnf'(R))\ [2\delta^k_{(i}R_{,j)} - g_{ij}g^{kl}R_{,l}]\ . \tag{32}$$

A comparison of (13) and (31) shows the difference between γ^{ij} and $\bar{g}^{ij}$. With the help of (32) and (31) we can rewrite the tracefree part of (29) in terms of the conformally related metric $\bar{g}_{ij}$

$$\begin{aligned}\bar{R}_{ab}(\bar{g}) - \frac{1}{4}\ \bar{R}(\bar{g})\ \bar{g}_{ab} = R_{ab}(g) - \frac{1}{4}\ R(g)\ g_{ab} - \frac{f''}{f'}[R_{,i;j} - \frac{1}{4}\ g_{ij}\Box R] - \\ -(\frac{f''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2)\ [R_{,i}R_{,j} - \frac{1}{4}\ g_{ij}R_{,l}R_{,m}g^{lm}]\ .\end{aligned} \tag{33}$$

When bringing the field equations into the form of Einstein's equations, the result is:

$$\begin{aligned}R_{ab}(g) - \frac{1}{2}\ R(g)\ g_{ab} = -2\kappa T_{ab}(g) - \frac{f''}{f'}[R_{,i;j} - g_{ij}\Box R] - \\ -[\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2]\ R_{,i}R_{,j} + (\frac{f'''}{f'} - \frac{3}{4}(\frac{f''}{f'})^2)\ g_{ij}R_{,l}R_{,m}g^{lm}\ .\end{aligned} \tag{34}$$

Again, the trace equation of (29), i.e.,

$$f'(R)R - 2f(R) = -2\kappa\ T_g\ , \tag{35}$$

is used to eliminate the curvature scalar in favour of the trace of the matter tensor. This means that the non-minimal coupling to the curvature scalar and its derivatives will be replaced by a coupling to the *gradients* of the matter variables contained in $g^{ik}T_{ik}$.

The remark at the end of section 3 for the case of covariant derivatives in the matter tensor applies here as well.

5 An example: Exponential gravity

5.1 New variational principle

As an example, we now take a recent model for $f(R)$-gravity [16] with:

$$f(R) = -cr(1 - e^{-\frac{R}{r}}) \ , f' = -ce^{-\frac{R}{r}} \ , \tag{36}$$

where r of dimension $(length)^2$ and c, dimensionless, are constants. From (14) $R_g = -\frac{1}{c}R_\gamma e^{\frac{R_\gamma}{r}}$. Thus, the inverse $R_\gamma = r^{-1}(R_g)$ can be obtained only numerically. A series expansion for $\frac{R_\gamma}{r} << 1$ leads to:

$$R_\gamma = -cR_g - \frac{c^2}{r}R_g^2 - \frac{3}{2}\frac{c^3}{r^2}R_g^3 \ \pm \cdots \ , \tag{37}$$

and

$$R_g = \frac{2\kappa T_g}{c^2} \ [1 - \frac{2\kappa T_g}{rc} + \frac{2}{3} \ (\frac{2\kappa T_g}{rc})^2 \pm \cdot\cdot] \tag{38}$$

If the further calculations are restricted *to the lowest order* in the expansion (37), with the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}R_g \ g_{ab}$ the field equations (20) become:

$$G_{ab}(\{^{k}_{ij}\}_g) = -\frac{2\kappa}{c^2} \ [1 - 4\frac{\kappa T_g}{rc}] \ T_{ab} \ - \frac{\kappa T_g}{c^2} \ \frac{\kappa T_g}{rc} \ g_{ab} \ . \tag{39}$$

For a perfect fluid with pressure $p = 0$, from (21) to lowest order the equations replacing Einstein's are:

$$R_{ab}(\{^{k}_{ij}\}_g) - \frac{1}{2}R_g \ g_{ab} = -\frac{2\kappa}{c^2}(1 - 4\frac{\kappa\mu}{rc}) \ \mu \ u_a u_b - \frac{\kappa^2\mu^2}{rc^3} \ g_{ab} \ . \tag{40}$$

The result is a variable coupling "constant" in the effective matter tensor and a variable cosmological term both depending on the energy density of matter.

For a homogeneous and isotropic cosmological model with scale factor $a(t)$ and flat space sections, (40) leads to altered Friemann equations:

$$(\frac{\dot{a}}{a})^2 = \frac{2\kappa\mu}{3c}\ (1 - \frac{7}{2}\frac{\kappa\mu}{cr})\ , \tag{41}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = \frac{\kappa^2\mu^2}{rc^3}\ . \tag{42}$$

Keep in mind that r, c are free constants of the model; the velocity of light has been put equal to 1 in (42). As numerical calculations would have to be done, and the main aim of this paper is the introduction of a new variational principle, we will not comment on this particular model (exponential gravity) and the physics following from it.

5.2 Palatini method

For exponential gravity as given by (36) and for pressureless fluid matter, the field equations according to the first equation of (29) turn out to be

$$R_{ik}(\Gamma) = \frac{2\kappa}{c}\ T_{ik}(g, u^A) + g_{ik}\ r(1 - e^{-\frac{R}{r}}). \tag{43}$$

This does not look complicated; however the connection Γ first must be expressed by the conformally related metric $\bar{g}_{ab}$. To the same order of approximation, the final field equation then can be written as:

$$R_{ab}(g) - \frac{1}{2}\ R(g)\ g_{ab} = -2\kappa\mu u_a u_b - \frac{2\kappa}{rc}[\mu_{,i;j} - g_{ij}\Box\mu] - \\ -\frac{2\kappa^2}{r^4c^2}\ \mu_{,i}\ \mu_{,j} + \frac{\kappa^2}{r^4c^2}\ g_{ij}\ \mu_{,l}\ \mu_{,m}\ g^{lm}\ . \tag{44}$$

The effective matter tensor in (44) depends on 1st and 2nd *gradients* of the energy density of matter. This shows that the physics resulting from the two variational principles may be quite different. The same can be said with regard to the Einstein-Hilbert metric variation used in [16] and e.g., in [17], [18] and the new variational principle.

6 Concluding remarks

For physics, a significant difference between the new variational method presented here and the Palatini method is that non-minimal coupling of matter and the curvature scalar R occurs by multiplication with functions of R or the trace of the matter tensor. In the Palatini method, non-minimal coupling happens via the *gradients* of the scalar curvature (trace of the matter tensor). A conceptual advantage of the new method is that it works within (pseudo)-Riemannian geometry; metric-affine geometry never does appear.[8] When dealing with $R + f(R)$-Lagrangians, in both approaches a new dimensionful constant is needed whose physical meaning must be defined. Application to $f(R,\ R_{ab}R^{ab})$ is unproblematic; here, two new parameters will occur. In general, via the field equations both curvature invariants can be expressed as functionals of invariants of the matter tensor. The Einstein-Hilbert Lagrangian seems to be very robust: now there are at least *three* different methods for a derivation of the Einstein field equations. As the example treated shows, for more general Lagrangians the variation will lead to different physical theories. Whether the new variational principle introduced here, if applied to cosmological models, produces convincing physics will have to be shown by further studies.

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