Palatini Variational Principle.

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Summary. — A detailed discussion of the Palatini variational principle is presented. It is shown how the Palatini variational principle can be derived from the more general Hilbert variational principle. The main purpose of this paper is to explain the origin of the Palatini variational principle in a logical way instead of stating it as an *ad hoc* result as is usual.

1. – Introduction.

The Palatini variational principle consists of varying the action for the free gravitational field with respect to independent variations of the metric g_{jk} and affine connection Γ^j_{kl} (1). The field equations that result from varying the metric are the vacuum Einstein equations

$$R^{jk} - \tfrac{1}{2} g^{jk} R = 0 . \tag{1.1}$$

The field equations that result from varying the affine connection imply that the affine connection must be equal to the Christoffel symbols $\begin{Bmatrix} j \\ k\ l \end{Bmatrix}$

$$\Gamma^j_{kl} = \begin{Bmatrix} j \\ k\ l \end{Bmatrix} = \frac{1}{2} g^{jh}(g_{hk,l} + g_{lh,k} - g_{kl,h}) . \tag{1.2}$$

The Palatini variational principle has a rather interesting history starting from the fact that PALATINI did not present in his paper what is now called

(1) C. MISNER, K. THORNE and J. WHEELER: *Gravitation* (San Francisco, Cal., 1973).

the Palatini variational principle (2). In one well-known reference the Palatini variational principle is described in a footnote as a « curious fact » (3).

The Palatini variational principle will also work for the nonvacuum case if only the metric and not the affine connection appears in the matter Lagrangian. In cases where the affine connection occurs in the matter Lagrangian (4) one uses the more general Hilbert variational principle (5). In the Hilbert variational principle one carries out variations with respect to the metric g_{jk} assuming the affine connection is equal to the Christoffel symbols. The resulting field equations are the Einstein equations. The Hilbert variational principle works for a general matter Lagrangian.

In this paper we shall present a detailed discussion of the Palatini variational principle and derive it from the Hilbert variational principle. Thus, we shall answer the questions: 1) Where does the Palatini variational principle come from? and 2) Why does the Palatini variational principle work?

2. – Hilbert variational principle.

The Lagrangian density for the free gravitational field is given by

$$L = c_1 \sqrt{-g}\, g^{jk} R_{jk}\,, \qquad c_1 = c^3/16\pi k\,, \tag{2.1}$$

where c is the speed of light, k the Newtonian gravitational constant and R_{jk} the Ricci tensor which is defined in terms of the affine connection Γ^j_{kl} by

$$R_{jk} = \Gamma^l_{jk,l} - \Gamma^l_{jl,k} + \Gamma^l_{nl}\Gamma^n_{jk} - \Gamma^l_{nk}\Gamma^n_{jl}\,. \tag{2.2}$$

In the Hilbert variational principle we wish to vary the action associated with the Lagrangian density (2.1) with respect to independent variations of the metric tensor g_{jk}. The metric tensor is symmetric, hence in variations with respect to the metric tensor the resulting equations will be symmetric. The simplest way to carry this out is to symmetrize the results after carrying out the variations.

The Hilbert variational principle starts from the same Lagrangian density

$$L = c_1 \sqrt{-g}\, g^{jk} R_{jk}\,. \tag{2.3}$$

In this variational principle we assume the affine connection is equal to the

(2) A. Palatini: *Rend. Circ. Math. Palermo*, **43**, 203 (1919).
(3) L. Landau and E. Lifshitz: *Classical Theory of Fields*, 3rd edition (Reading, Mass., 1971), p. 273.
(4) One example where this occurs is for a perfect fluid, see for example J. Ray: *Journ. Math. Phys.*, **13**, 1453 (1972).
(5) D. Hilbert: *Gottinger Nachr.* (1915).

Christoffel symbols as given in eq. (1.2). The variational derivative with respect to the metric is defined by

$$\frac{\delta L}{\delta g_{jk}} = \frac{\partial L}{\partial g_{jk}} - \left(\frac{\partial L}{\partial g_{jk,r}}\right)_{,r} + \left(\frac{\partial L}{\partial g_{jk,r,s}}\right)_{,s,r} . \tag{2.4}$$

Carrying out the variation of the action associated with (2.3) with respect to g_{jk} we find the field equations

$$\frac{\delta L_1}{\delta g_{jk}} = - c_1 \sqrt{-g}\left(R^{jk} - \frac{1}{2} g^{jk} R\right) = 0 . \tag{2.5}$$

3. – Palatini variational principle.

In the Palatini variational principle we wish to vary the action associated with (2.3) with respect to g_{jk} and Γ^j_{kl} independently. Hence, we must add to (2.3) the connection between Γ^j_{kl} and the Christoffel symbols $\left\{ {j \atop k\, l} \right\}$ as a constraint. We shall assume the connection is symmetric. We then have the Lagrangian density

$$L_2 = c_1 \sqrt{-g}\, g^{jk} R_{jk} + \lambda^{kl}_j \left(\Gamma^j_{kl} - \left\{ {j \atop k\, l} \right\}\right) . \tag{3.1}$$

We must now vary the action associated with L_2 with respect to g_{jk}, Γ^j_{kl} and λ^{kl}_j. Carrying out the variations associated with Γ^j_{kl} and λ^{kl}_j results in

$$\begin{aligned} &\delta^l_j g^{rs} \Gamma^k_{rs} + g^{kl} \Gamma^r_{jr} - g^{rl} \Gamma^k_{rj} - g^{kr} \Gamma^l_{jr} - \\ &\quad - \tfrac{1}{2} \delta^r_j g^{kl} g^{sp} g_{sp,r} + \tfrac{1}{2} \delta^l_j g^{kr} g^{sp} g_{sp,r} + g^{kr} g^{ls} g_{rs,j} - \delta^l_j g^{kr} g^{ps} g_{rs,p} + \lambda^{kl}_j = 0 , \end{aligned} \tag{3.2}$$

$$\Gamma^j_{kl} - \left\{ {j \atop k\, l} \right\} = 0 . \tag{3.3}$$

Note that one may symmetrize (3.2) with respect to k, l, but this will not change any of the final results. From (3.3) it follows that Γ^j_{kl} is equal to the Christoffel symbol. Using this and the symmetry of the metric in (3.2) we find

$$\lambda^{kl}_j = 0 . \tag{3.4}$$

Now carrying out the variations with respect to the metric using (3.4) we find

$$R^{jk} - \tfrac{1}{2} g^{jk} R = 0 , \tag{3.5}$$

the vacuum Einstein equations. If we use the result (3.4) in L_2, then the constraint term vanishes. This means we can carry out the variation neglecting the constraints and obtain the same results as with the constraints present. This is the same type of result we have recently discussed when we change from configuration-space variations to phase-space variations in classical mechanics [6]. This means we can vary the action associated with

$$L = c_1 \sqrt{-g}\, g^{jk} R_{jk} \tag{3.6}$$

with respect to independent variations of g_{jk} and Γ^j_{kl} and obtain the Einstein vacuum equations and the fact that the affine connection must be equal to the Christoffel symbols. Variation of the action associated with (3.6) with respect to g_{jk} yields the vacuum Einstein equations as before. Variation with respect to Γ^j_{kl} yields (3.2) with λ^{kl}_j set to zero:

$$\begin{aligned}\delta^l_j g^{rs} \Gamma^k_{rs} + g^{kl} \Gamma^r_{jr} - g^{rl} \Gamma^k_{rj} - g^{kr} \Gamma^l_{jr} - \\ - \tfrac{1}{2} \delta^r_j g^{kl} g^{sp} g_{sp,r} + \tfrac{1}{2} \delta^l_j g^{kr} g^{sp} g_{sp,r} + g^{kr} g^{ls} g_{rs,j} - \delta^l_j g^{kr} g^{ps} g_{rs,p} = 0\,.\end{aligned} \tag{3.7}$$

Now if we use the fact that g_{jk} and Γ^j_{rl} are symmetric it follows from (3.7) that Γ^j_{kl} is equal to the Christoffel symbols.

4. – Conclusions.

We have presented a detailed discussion of the Palatini variational principle which derives it from the more general Hilbert variational principle. What we have found is that, if in the Hilbert principle we drop the assumption that the affine connection is equal to the Christoffel symbols and add this condition to the Lagrangian via Lagrange multipliers, then from the field equations we find that the Lagrange multipliers associated with the constraints are zero. This means that the constraint does not need to be considered in carrying out the variations. However, if we drop the constraint from the Lagrangian and carry out independent variations with respect to the metric and the affine connection, this is exactly the Palatini variational principle as it is usually presented.

The purpose of this paper has been to clarify the Palatini variational principle. We hope that this discussion will make it unnecessary to refer to the Palatini variational principle as a « curious fact ».

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(6) J. RAY: *Am. Journ. Phys.*, **41**, 1188 (1974).

● RIASSUNTO (*)

Si presenta una discussione dettagliata del principio variazionale di Palatini. Si mette in evidenza come si possa ottenere il principio variazionale di Palatini dal più generale principio variazionale di Hilbert. Ciò che principalmente ci si propone con questo articolo è spiegare l'origine del principio variazionale di Palatini in maniera logica piuttosto che emunciarlo come un risultato *ad hoc* come si fa di solito.

(*) *Traduzione a cura della Redazione.*

Вариационный принцип Палатини.

Резюме (*). — Предлагается подробное обсуждение вариационного принципа Палатини. Показывается, как вариационный принцип Палатини может быть выведен из более общего вариационного принципа Гильберта. Основная цель этой статри — объяснить происхождение вариационного принципа Палатини логическим образом, вместо того, чтобы доказывать его для данного конкретного случая, как обычно.

(*) *Переведено редакцией.*