Albert Einstein's 1916 Review Article on General Relativity[1]

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Abstract

The first comprehensive overview of the final version of the general theory of relativity was published by Einstein in 1916 after several expositions of preliminary versions and latest revisions of the theory in November 1915. A historical account of this review paper is given, of its prehistory, including a discussion of Einstein's collaboration with Marcel Grossmann, and of its immediate reception.

First publication. *Annalen der Physik, 49*, 769–822; also published separately as Leipzig: Barth, 1916.

Later editions. Various reprints of the separately printed version, 5th unaltered reprint in 1929; included in the 3rd and later editions of the anthology Lorentz, H.A., Einstein, A., Minkowski, H., *Das Relativitätsprinzip*, Leipzig: Teubner, 1919[3]; in the 1960 Readex Microprint edition of the *Published Writings of Albert Einstein* as item 78; in the anthology K.v. Meyenn (ed.), *Albert Einstein's Relativitätstheorie. Die grundlegenden Arbeiten*, Braunschweig: Vieweg, 1990; first edition reprinted with annotation as Doc. 30 in Vol. 6 of the *Collected Papers of Albert Einstein*, Princeton: Princeton University Press, 1996, pp. 283–339; the German and English reprints of the *Collected Papers* are also available online at http://www.alberteinstein.info (2003).

English translations. 1920 by S. N. Bose, in: *The Principle of Relativity. Original Papers by A. Einstein and H. Minkowski*, Calcutta: University of Calcutta Press, pp. 90–163; 1923 by W. Perrett and G. B. Jeffery (without the first page),

[1]To appear in: *Landmark Writings in Western Mathematics, 1640–1940*, Ivor Grattan-Guiness (ed.), Elsevier 2004.

in English edition of Lorentz, H. A. et al., *The Principle of Relativity*, London: Methuen 1923 (Dover reprint 1952), pp. 111–164.

French translations. 1933 by M. Solovine, in: Albert Einstein. *Les Fondements de la Théorie de la Relativité Générale. Théorie unitaire de la Gravitation et de l'Electricité. Sur la Structure Cosmologique de l'Espace.* Paris: Hermann, pp. 7–71; 1993 by F. Balibar et al., in *Albert Einstein. Oeuvres choisies*, tome 2, Paris: Éditions du Seuil, Éditions du CNRS, pp. 179–227.

Russian translation. 1935, in *Albert Einstein. Sobranie naychnykh trudov*, vol. 1, Moscow: Izdatel'stvo 'Nauka', 1965, pp. 452–504.

Spanish translation. 1950 by F. Alsina Fuertes and D. Canals Frau, in Albert Einstein. *La relatividad (Memorias originales)*, Buenos Aires: Emecé editores, pp. 115–223.

Manuscripts. A manuscript of 46 pages is in the Schwadron collection at the Hebrew University Jerusalem, available online at http://www.alberteinstein.info under Call Nr. 120-788.

1. The special theory of relativity

Some ten years before the first review of the *general* theory of relativity, Einstein published his famous paper *On the Electrodynamics of Moving Bodies* (Einstein 1905). That paper introduced what later became to be called the *special* theory of relativity. It presented a conceptual analysis of the notions of space and time, with a critical reassessment of the meaning of simultaneity at its core. Length contraction and time dilation in a system that is in uniform relative motion to an observer with a speed comparable to that of light are its most salient features.

The 1905 paper was not a very sophisticated paper on the mathematical side. Its author had obtained a diploma as secondary school teacher for mathematics and physics at the Polytechnic Zurich in 1900 (Pais 1982), (Fölsing 1998). His science education had been excellent with laboratory work in the most up-to-date facilities and first-rate mathematics teachers, like Adolf Hurwitz (1859–1919), Carl Friedrich Geiser (1843–1934), and Hermann Minkowski (1864–1909). If more recent advances in theoretical physics were somewhat neglected by his physics teacher Heinrich Friedrich Weber (1843–1912), the young Einstein made up for it in extensive autodidactic studies. Fascinated by laboratory experience, Einstein seems to have skipped more than one of his mathematics lectures, though, and obtained his knowledge when preparing for examinations with the help of lecture notes that had been carefully worked out by his more mathematically inclined friend Marcel Grossmann (1878–1936).

After initial attempts to start a traditional academic career had failed, Einstein composed his theory of special relativity in the evening hours after office work as a technical expert, especially for electrotechnology, at the Patent office in Bern. Mathematically, the breakthrough of special relativity came in a representation using only standard techniques of elementary calculus. Maxwell's electromagnetic equations were written component-wise, notwithstanding the fact, that compact vector notation had already been well developed, if not standardized, in electrodynamics and hydrodynamics by the end of the nineteenth century.

The subsequent generalization of the special theory of relativity to a generally covariant theory of gravitation proceeded in three major steps (Norton 1984), (Stachel 1995), (Renn and Sauer 1999), (Stachel 2002, sec. V), (Renn et al. forthcoming). For further references, see the literature cited in these works, and, on specific aspects, see also volumes 1 (Howard and Stachel 1989), 3 (Eisenstaedt and Kox 1992), 5 (Earman, Janssen and Norton 1995), and 7 (Goenner et al. 1999) of the Einstein Studies series. These steps are:

- the formulation of the *equivalence hypothesis* in 1907,
- the introduction of the *metric tensor* as the crucial mathematical concept for a generally relativistic theory of gravitation in 1912,
- and the discovery of the generally covariant *field equations of gravitation* in 1915.

2. The equivalence hypothesis

In 1907, Einstein saw himself confronted with the task of reflecting on the consequences of the relativity principle for the whole realm of physics. He was asked to write a review article *On the Relativity Principle and the Conclusions Drawn from It* (Einstein 1907). The reinterpretation of the concept of simultaneity in special relativity was hinging on the finiteness of the speed of light for signal transmission. It was therefore clear that the Newtonian theory of gravitation posed an embarrassment. In Newtonian mechanics, the gravitational force is an action-at-a-distance force and thus contradicts the fundamental assumption of special relativity that no physical effects can propagate with a speed superseding a finite value. In reflection on this difficulty, Einstein took a decisive turn. He linked the problem of the instantaneous propagation of the gravitational force in Newtonian physics to the problem of generalizing the principle of (special) relativity to non-uniform relative motion. In a reinterpretation of Galileo's law of free fall, according to which all bodies in a gravitational field undergo the same acceleration regardless of their weight, Einstein formulated the so-called equivalence hypothesis. According to this hypothesis, there is no conceivable experiment that could distinguish between processes taking place in a static and homogeneous gravitational field and those that are only viewed from a frame of reference that is uniformly and rectilinearly accelerated in a gravitation free space. The value of this hypothesis was a heuristic one. It enabled Einstein to investigate the effects of gravitation in a relativistic theory by analyzing the corresponding processes if interpreted from an accelerated frame of reference.

Already in 1907, Einstein drew three important consequences from the equivalence hypothesis. He concluded that the time and hence also the speed of light must depend on the gravitational potential. Consequently, the frequency of light emitted from the sun should be shifted towards the red, and light rays passing through a gravitational field would be bent. He also concluded that every energy should have not only inertial but also gravitational mass.

Incidentally, this is also the time when Einstein began to use the term 'relativity theory' (*Relativitätstheorie*) in print, e.g. (Einstein 1907, p. 439). The term had first been used in print in the same year by Paul Ehrenfest (1880–1933), after Max Planck (1858–1947) had earlier introduced the term *Relativtheorie*. A suggestion by Felix Klein (1849–1925) in 1910, to use the perhaps more appropriate term 'invariant theory' (*Invariantentheorie*) was not taken up (*The Collected Papers of Albert Einstein* (CPAE), Vol. 2, p. 254).

While the equivalence hypothesis of 1907 provided a point of departure for a generalization of the theory of relativity and for a new field theory of gravitation, Einstein did not present a solution to the problem of instantaneous propagation of the gravitational force. While Einstein remained rather silent on the topic of the relativity principle for some years, these questions were taken up by others. Hermann Minkowski and Henri Poincaré, e.g., proposed Lorentz-covariant generalizations of Newton's law of gravitation. More importantly, Minkowski also gave the theory of relativity a more sophisticated mathematical representation. Reflecting on the symmetry of the Lorentz transformations, Minkowski used elements from Cayley's matrix calculus to give the equations a four-dimensional representation and to interpret the Lorentz transformations as rotations in a four-dimensional vector space (Minkowski 1908). In a report of his work to the 80th general assembly of physicians and scientists in Cologne, he illustrated this interpretation by the often-quoted words:

> From this hour on, space by itself, and time by itself, shall be doomed to fade away in the shadows, and only a kind of union of the two shall preserve an independent reality. (Minkowski 1909, 105)

Minkowski's four-dimensional representation was taken up by Arnold Sommerfeld (1868–1951) who developed a four-dimensional vector algebra and vector calculus and by Max Laue (1879–1960) who focused upon the tensorial representation of the stress-energy-momentum complex.

3. The metric tensor

Einstein resumed work on the subject again in 1911. By then he had been appointed ordinary professor of physics at the German university in Prague. In a series of papers, he developed a theory of the static gravitational field, following the heuristics of the equivalence assumption of static homogeneous gravitational fields to systems in uniform and rectilinear acceleration (Einstein 1911), (1912a), (1912b). His work was boosted by a competition with Max Abraham (1875–1922) who had picked up on Einstein's idea of a variable speed of light and had suggested a dynamic theory of gravitation. Abraham had proposed a field equation where the d'Alembertian acting on the speed of light c was proportional to the scalar mass density. In the course of the debate it quickly became clear that with variable c Abraham's equation was Lorentz covariant at best in some ill-defined infinitesimal sense and could hardly be interpreted consistently. But Abraham had demonstrated to Einstein the technical power of a four-dimensional representation, and had prepared him to take the second big step of introducing the metric tensor.

The second indication of where to go next in the course of generalizing the relativity principle came from the analysis of rotating frames of reference. The heuristic assumption of the equivalence hypothesis implied that also centrifugal and Coriolis forces should be interpreted as gravitational forces. Looking at the invariant $c^2dt^2 - (dx^2 + dy^2 + dz^2)$ in rotating frames of reference would produce terms of the form $2\omega dx'dt'$ where the angular velocity ω would have to be interpreted as a gravitational potential, just as in the theory of static gravitation the speed of light $c = c(x, y, z)$ had assumed the role of a variable gravitational potential. Since moreover the measuring rods for determining the circumference, but not the diameter, of a rotating disk are Lorentz contracted, the analysis of a rotating disk already pointed to a breakdown of Euclidean geometry.

4. Einstein's collaboration with Marcel Grossmann

At some point around this time, Einstein remembered Geiser's lectures on Gaussian surface theory which he had studied through his friend's Grossmann's notes. It occurred to him that the invariant line element of differential geometry might be the key to finding a proper mathematical representation for his problem. Fortunately, Einstein had just accepted a call to the Zurich Polytechnic where Grossmann had become professor of geometry in 1907. Einstein asked Grossmann for help in studying the mathematical literature, and the two embarked on an intense collaboration. About this collaboration, he wrote in October 1912:

> I am now working exclusively on the gravitation problem and believe that I can overcome all difficulties with the help of a mathematician friend of mine here. But one thing is certain: never before in my life have I troubled myself over anything so much, and I have gained enormous respect for mathematics, whose more subtle parts I considered until now, in my ignorance, as pure luxury. (CPAE, Vol. 5, Doc. 421).

The question that Einstein put to Grossmann was to identify the mathematics connected with the invariance of a four-dimensional infinitesimal line element with metric tensor $g_{\mu\nu}$

$$ds^2 = \sum_{\mu\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

A research notebook with calculations from that time documents Einstein's and Grossmann's cooperation (Norton 1984), (Renn and Sauer 1999), (Renn et al. forthcoming). It is in this so-called 'Zurich notebook', that we find the first written instance of the metric tensor for (3+1)-dimensional space-time (Renn and Sauer 1999, 96), see also Call No. 3-006, image 39, on http://www.alberteinstein.info (2003) for a facsimile. Realizing that the vector calculus for Euclidean space in curvilinear coordinates is formally equivalent to the calculus of a general manifold equipped with an invariant infinitesimal line element, Grossmann saw that the task was to generalize the four-dimensional vector calculus developed by Minkowski, Sommerfeld, Laue, and others using methods of an altogether coordinate independent calculus. Scanning the literature, Grossmann soon found the necessary mathematical concepts in (Riemann 1892) on n-dimensional manifolds, in (Christoffel 1869) on quadratic differential forms, and in (Ricci and Levi-Civita 1901) on their so-called absolute differential calculus.

It seems that Einstein and Grossmann quickly saw how to formulate, in outline, a generally covariant theory with the metric tensor $g_{\mu\nu}$ representing the gravito-inertial field. In the following discussion, I will give all formulas in a notation that is both slightly modernized and made consistent over the various texts discussed. In particular, I will abbreviate coordinate derivatives by subscript commas, use the Einstein summation convention of summing over repeated indices, and denote functional derivatives by δ rather than ∂. In their joint publications, Einstein and Grossmann also used Greek letters to denote contravariant vectors and tensors rather than superscript indices.

Einstein and Grossmann found generally covariant equations of motion of a material point of invariant mass m for a given metric field $g_{\mu\nu}$ in the absence of non-gravitational forces as

$$\delta \left\{ \int L dt \right\} = \delta \left\{ -m \int ds \right\} = 0, \tag{2}$$

with a particle Lagrangian $L = -m ds/dt$. In a generalization to a continuous distribution of matter characterized by an energy-momentum tensor for pressureless flow of dust with rest mass density ρ_0,

$$T^{\mu\nu} = \rho_0 \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}, \tag{3}$$

the equation of motion turned into ($g = \det(g_{\mu\nu})$)

$$\left(\sqrt{-g} g_{\sigma\mu} T^{\mu\nu} \right)_{,\nu} - \frac{1}{2} \sqrt{-g} g_{\mu\nu,\sigma} T^{\mu\nu} = 0. \tag{4}$$

The latter equation is an explicit expression for the vanishing of the covariant divergence of the mixed tensor density $\sqrt{-g} T_\sigma^\nu$. It is as such closely related to the conservation of energy-momentum as can be seen by integrating $T^{\mu\nu}$ over a closed 3-surface and invoking Gauss's theorem. In Einstein's interpretation, the first term of (4) gave the conservation law for special relativity for constant $g_{\mu\nu}$, and the second part consequently represented the energy-momentum flow due to the gravitational field. This interpretation led Einstein to believe that the gravitational force components are given by $g_{\mu\nu,\sigma}$. The task remained to find a field equation for the metric tensor field, i.e. a tensorial generalization of the Poisson equation.

5. Coming close to the solution, or so it seems

From Riemann's and Christoffel's investigations, Grossmann and Einstein learned that the crucial mathematical concept was the Riemann curvature tensor $\{ik, lm\}$ given in terms of the Christoffel symbols of the second kind (given in the old-fashioned notation),

$$\left\{\begin{matrix}\mu\ \nu\\ \tau\end{matrix}\right\} = g^{\tau\lambda}\left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right), \tag{5}$$

as

$$\{\iota\kappa, \lambda\mu\} = \left\{\begin{matrix}\iota\ \lambda\\ \kappa\end{matrix}\right\}_{,\mu} - \left\{\begin{matrix}\iota\ \mu\\ \kappa\end{matrix}\right\}_{,\lambda} + \left\{\begin{matrix}\iota\ \lambda\\ \rho\end{matrix}\right\}\left\{\begin{matrix}\rho\ \mu\\ \kappa\end{matrix}\right\} - \left\{\begin{matrix}\iota\ \mu\\ \rho\end{matrix}\right\}\left\{\begin{matrix}\rho\ \lambda\\ \kappa\end{matrix}\right\}, \tag{6}$$

(see Figure 1). Since the right-hand side of the field equation would be given by the stress-energy tensor of matter, a tensor of second rank, the left-hand side of the field equation also had to to be a two-index object. But the obvious candidate, the Ricci tensor

$$R_{\mu\nu} = \{\mu\kappa, \kappa\nu\}, \tag{7}$$

would not produce a field equation that was acceptable to Einstein and Grossmann at the time. Although a field equation,

$$R_{\mu\nu} + \kappa T_{\mu\nu} = 0, \tag{8}$$

with some constant κ was considered as a candidate, they dismissed it because they were unable to recover familiar Newtonian physics in the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu} = \mathrm{diag}(1, 1, 1, \text{-}1)$, $|h_{\mu\nu}| \ll 1$, and $|h_{\mu\nu,\rho}| \ll 1$.

The dismissal of the candidate (8) has been a major puzzle for historians for a long time. Since in the vacuum case, $T_{\mu\nu} \equiv 0$, (8) is equivalent to the final field equations of general relativity (see (22) below), Einstein and Grossmann had come by a hair's breadth to arriving at general relativity already at this point, or so it seems. However, a closer analysis of the Zurich notebook revealed that Einstein had to overcome more conceptual difficulties before he was ready to accept a generally covariant theory (Renn and Sauer 1999), (Renn et al. forthcoming).

6. The *Entwurf* theory

After giving up the attempt to base a field equation on the Riemann curvature tensor, Einstein and Grossmann constructed a field equation that was closer to their heuristic requirements of energy conservation and recovery of the

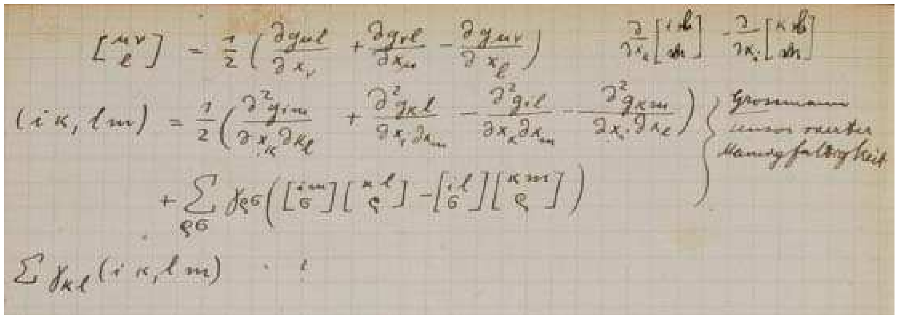

Figure 1: Top portion of p. 14L of the 'Zurich Notebook' (Einstein Archives Call.No. 3-006). Next to Grossmann's name Einstein writes down the Christoffel symbols of the first kind, and the fully covariant Riemann tensor (ik, lm) which he calls a "tensor of fourth manifold" (*Tensor vierter Mannigfaltigkeit*). Einstein then begins to investigate the Ricci tensor by contracting with the contravariant metric γ_{kl}. © The Hebrew University of Jerusalem, Albert Einstein Archives. Reproduced with permission.

Poisson equation in the Newtonian limit. The idea was to take the expression $\left(g^{\alpha\beta}g^{\mu\nu}_{,\beta}\right)_{,\alpha}$ which would clearly reduce to the d'Alembertian and Laplacian operators in the weak field and static limits and substitute it for $T^{\mu\nu}$ in the second term of (4). If additional terms of higher order could be identified such that this expression could be transformed into a total divergence, energy-momentum conservation in the form of (4) would automatically be satisfied. The field equations they found read

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}_{,\beta}\right)_{,\alpha} - g^{\alpha\beta}g_{\tau\rho}g^{\mu\tau}_{\alpha}g^{\nu\rho}_{\beta} + \\ +\frac{1}{2}g^{\alpha\mu}g^{\beta\nu}g_{\tau\rho,\alpha}g^{\tau\rho}_{,\beta} - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}g_{\tau\rho,\alpha}g^{\tau\rho}_{,\beta} = -\kappa T^{\mu\nu}. \quad (9)$$

In early summer 1913, Einstein and Grossmann proceeded to publish their findings in a little booklet under the title *Outline* [Entwurf] *of a Generalized Theory of Relativity and of a Theory of Gravitation* (Einstein and Grossmann 1913). As the title page indicated, it was divided into two parts, a physical part for which Einstein signed responsible, and a mathematical part for which Grossmann signed as author.

The *Entwurf* theory, as it is frequently called in modern historical literature, was a hybrid theory, if viewed from our modern understanding of general relativity. It presented a mathematical apparatus of tensor calculus that allowed to formulate a theory in a generally covariant manner, and it gave generally covariant equations of motion. Just as in the final theory of general relativity, the crucial concept was the metric tensor which was interpreted as representing a gravito-inertial field. All these elements were later to be found in the final version of general relativity. The only thing that was missing were generally covariant field equations.

The hybrid character of the *Entwurf* theory is reflected in a certain ambivalence that Einstein showed with respect to their achievement. Initially and also again and again over the following two years he expressed himself rather pleased with the theory. He had settled on the *Entwurf* equations as acceptable equations and began to elaborate their consequences. From an unpublished manuscript we know that together with his friend Michele Besso (1873–1955) he calculated the advance of the planetary perihelia. For Mercury, it was well known that the observed perihelion advance was in discrepancy with the value calculated on the basis of Newtonian mechanics, and this anomaly was the most prominent quantitative failure of classical gravitation theory. Not surprisingly, they found a value for Mercury that was significantly off the observed value: theirs even came with the wrong sign (Earman and Janssen 1993).

Notwithstanding Einstein's acceptance of the *Entwurf* equations, he also indicated that the restricted covariance of these equations was a 'black spot' of the theory. His initial heuristics clearly did not imply any reason for a restricted covariance of the theory. In further reflection, Einstein convinced himself, however, that this restricted covariance was, in fact, to be expected. He devised an argument to the effect that indeed no generally covariant field equations were physically admissable. The argument was first published in an addendum to a reprint of the *Entwurf* in the *Zeitschrift für Mathematik und Physik*.

He considered a hole in four-dimensional space-time, i.e. a finite region with vanishing stress-energy $T_{\mu\nu} \equiv 0$. Let $G(x)$ denote a solution $g_{\mu\nu}(x_1, x_2, x_3, x_4)$ of the field equations, and perform a coordinate transformation within the hole, i.e. consider a coordinate system x' that coincides smoothly with the original coordinate system x at the boundary of the hole. In the primed coordinates the transformed field $G'(x')$ is the solution to the transformed field equations. But if the field equations are generally covariant, then $G'(x)$ is also a solution to the original field equations. We hence arrive at two distinct solutions in the same coordinate system x for the same distribution of matter $T_{\mu\nu}$. Einstein concluded that generally covariant field equations cannot uniquely determine the physical processes in a gravitational field. Consequently, one had to restrict the admissible coordinate systems to what he began to call 'adapted coordinates'.

Already in their *Entwurf*, Einstein and Grossmann had stated that the most urgent unsolved problem of their theory was the identification of the covariance group of their field equations. The solution to this question was made possible by a variational reformulation of the theory. It was the topic of Einstein's and Grossmann's second joint publication (Einstein and Grossmann 1914).

As acknowledged in a footnote, the hint of trying a variational approach came from Paul Bernays (1888–1977), a student of David Hilbert (1862–1943) in Göttingen. The idea was that a variational formulation might help to identify the group of 'adapted coordinates' since it would be easier to identify the invariance group of the scalar action integral than the covariance group of the explicit tensorial field equations. Einstein and Grossmann indeed succeeded to cast the *Entwurf* theory in a variational formulation,

$$\delta \left\{ \int L d^4 x \right\} = 0, \tag{10}$$

with a Lagrangian

$$L = \sqrt{-g} \left(\frac{1}{4} g^{\alpha\beta} g_{\tau\rho,\alpha} g_{\beta}^{\tau\rho} - \kappa L^{(\mathrm{mat})} \right), \tag{11}$$

where the matter part $L^{(\mathrm{mat})}$ was not included explicitly.

Considering variations adapted to the hole consideration, they were now able to identify the condition for 'adapted coordinates' governing the covariance group of the *Entwurf* as

$$B_\sigma = \left(\sqrt{-g} g^{\alpha\beta} g_{\sigma\mu} g^{\mu\nu}_{,\beta} \right)_{,\nu\alpha} = 0. \tag{12}$$

With their second joint paper, the collaboration between Einstein and Grossmann came to an end. In spring 1914, Einstein moved to Berlin taking up a position as member of the Prussian Academy of Sciences in Berlin, a move which relieved him of his teaching load as professor at the Zurich polytechnic.

7. The 1914 review article on the *Entwurf* theory

In summer 1914, Einstein felt that the new theory should be presented in a comprehensive review. He also felt that a mathematical derivation of the field equations that would determine them uniquely was still missing.

Both tasks are addressed in a long paper, presented in October 1914 to the Prussian Academy for publication in its *Sitzungsberichte* (Einstein 1914). It is entitled *The Formal Foundation of the General Theory of Relativity* and Einstein thus, for the first time, gave the new theory of relativity the epithet 'general' in lieu of the more cautious 'generalized' that he had used for the *Entwurf*.

The paper is divided into five sections, and thus anticipates the structure of the final 1916 review. An introductory section on the basic ideas of the theory is followed by a section on the theory of covariants. This section replaced Grossmann's mathematical part of the joint *Entwurf* paper and gives an account of the elements of tensor calculus employed in the theory. A third section discusses the theory for a given metric field. It introduced the stress-energy-momentum tensor and discussed the conservation laws associated with the vanishing of its divergence, as well as the equations of motions and the electromagnetic field equations.

The fourth section gave a new derivation of the *Entwurf* equations. Einstein here tried to give a derivation that supposedly rendered them unique. He reiterated the hole consideration and introduced adapted coordinates. The variation is now done in a generic manner for the gravitational part H of the Lagrangian L. In order to fix the Lagrangian, Einstein assumes H to be a homogeneous function of second degree in the coordinate derivatives $g^{\mu\nu}_{,\sigma}$ of the metric, and picks from the allowed combinations the one that conforms to the adapted coordinate condition.

In a final, short section Einstein discussed approximations of the theory, recovered the Newtonian limit and predicted both gravitational light bending and red shift.

8. The demise of the *Entwurf* and the breakthrough to general covariance

Einstein had known that the *Entwurf* equations produced the wrong perihelion advance for Mercury since 1913. A second set-back that undermined his confidence in the theory came in spring 1915 when Levi-Civita carefully studied Einstein's long Academy paper and found fault with its derivation of the field equations. After an intense epistolary exchange in March and April 1915, Einstein had to admit that his proof of the tensorial character of the left hand side of the field equations for admissible coordinate transformations was incomplete (CPAE, Vol. 8, Doc. 80).

In September 1915, Einstein realized that the Minkowski metric in rotating Cartesian coordinates is not a solution to the *Entwurf* equations. Earlier checks of this condition appear to have been flawed by trivial algebraic mistakes that conspired to convince Einstein of the validity of this heuristic requirement (Janssen 1999).

The final blow came quickly afterwards when Einstein discovered that the alleged uniqueness of the field equations in his derivation of the Academy paper did not hold up.

At this point, Einstein began to reconsider alternatives for the gravitational field equations. He reflected on considerations that he had done previously in his search for the *Entwurf* equations. A closer analysis of the Zurich notebook indeed revealed that in the fall of 1915, Einstein reconsidered the same candidates for field equations as he had done in 1912 (Norton 1984), (Renn and Sauer 1999), (Renn et al. forthcoming). The return to general covariance is documented in four communications to the Prussian Academy, presented on November 4, 11, 18, and 25, and each published a week later in the *Sitzungsberichte*.

In the first communication, Einstein announced that he had lost his faith in the *Entwurf* equations and wrote

> In this pursuit I arrived at the demand of general covariance, a demand from which I parted, though with a heavy heart, three years ago when I worked together with my friend Grossmann. As a matter of fact, we were then quite close to that solution of the problem, which will be given in the following. (Einstein 1915a, 778)

Einstein now split the Ricci tensor into two parts,

$$R_{\mu\nu} = \{\mu\kappa, \kappa\nu\} = N_{\mu\nu} + M_{\mu\nu}, \tag{13}$$

where

$$N_{\mu\nu} = -\left\{\begin{matrix}\mu\ \nu\\ \kappa\end{matrix}\right\}_{,\kappa} + \left\{\begin{matrix}\mu\ \kappa\\ \rho\end{matrix}\right\}\left\{\begin{matrix}\rho\ \nu\\ \kappa\end{matrix}\right\}, \tag{14}$$

and

$$M_{\mu\nu} = -\left\{\begin{matrix}\mu\ \kappa\\ \kappa\end{matrix}\right\}_{,\nu} + \left\{\begin{matrix}\mu\ \nu\\ \rho\end{matrix}\right\}\left\{\begin{matrix}\rho\ \kappa\\ \kappa\end{matrix}\right\}. \tag{15}$$

Since $\left\{\begin{matrix}\mu\ \kappa\\ \kappa\end{matrix}\right\} = (\ln\sqrt{-g})_{,\mu}$ is a vector for all transformations that leave g invariant (unimodular substitutions), $M_{\mu\nu}$ is a covariant derivative of a vector, and hence all quantities in (13) are tensors under such substitutions.

The field equations of the first November communication were now given as

$$N_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{16}$$

Even though Einstein explicitly reverted to the general covariance of the Riemann-Christoffel tensor, the field equations of the first November communication are not generally covariant but only covariant under unimodular coordinate transformations.

The restricted covariance is immediately obvious also from the variational formulation that Einstein provided. Looking again at the geodesic equation,

$$\frac{d^2x^\tau}{ds^2} + \left\{ \begin{matrix} \mu\ \nu \\ \tau \end{matrix} \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{17}$$

as the equation of motion for a point particle in a given gravitational field, Einstein now conceived of the negative Christoffelsymbols $\Gamma^\sigma_{\mu\nu} = -\left\{ \begin{matrix} \mu\ \nu \\ \sigma \end{matrix} \right\}$ as the components of the gravitational force rather than the simple coordinate derivatives of the metric $g_{\mu\nu,\sigma}$. These quantities now entered into the gravitational part of the Lagrangian as

$$L = g^{\sigma\tau}\Gamma^\alpha_{\sigma\beta}\Gamma^\beta_{\tau\alpha} - \kappa L^{(\mathrm{mat})}. \tag{18}$$

(cp. (11). He observed that weak fields now allow to go to the Newtonian limit, and that the transition to rotating frames of reference is admissible since the corresponding coordinate transformations have unit determinant.

Not only was the covariance of the theory restricted to unimodular transformations, Einstein also showed that energy-momentum conservation demanded that a coordinate restriction,

$$\left(g^{\alpha\beta}\left[\ln\sqrt{-g}\right]_{,\beta}\right)_{,\alpha} = -\kappa T, \tag{19}$$

had to be satisfied. Since, in general, the trace of the energy-momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$ does not vanish, (19) implies that coordinates cannot be chosen arbitrarily. In particular, (19) implies that one cannot set $\sqrt{-g} \equiv 1$.

At this point, it needs to be mentioned that Einstein's return to general covariance in November 1915 was done in a hasty competition with Hilbert (Sauer 1999). Einstein had given a series of lectures on the *Entwurf* theory in Göttingen earlier in the summer, and Hilbert had then closely studied Einstein's theory over the fall. Apparently, Hilbert had found fault with Einstein's derivation of the field equations, too, and Einstein had heard about Hilbert's criticism through Sommerfeld (CPAE, Vol. 8, Doc. 136). When he received proofs of his first November communication, he forwarded them to Göttingen, and it seems that Hilbert responded immediately with a report about his own progress. Hilbert, at the time, believed in an electromagnetic world-view and had been working on combining Einstein's gravitational theory with a generalized version of Maxwellian electrodynamics suggested by Gustav Mie (1868–1957). Mie had proposed a theory of matter where non-linear, but Lorentz-covariant generalizations of Maxwell's equations should allow for particle-like solutions in the microscopic realm. It seems likely that Hilbert had informed Einstein about the basic characteristics of his approach which aimed at a unification of Einstein's and Mie's theories.

The second of Einstein's four famous November communications, in any case, discussed the possibility of a purely electromagnetic origin of matter (Einstein 1915b). Since in classical electromagnetism, the stress-energy-momentum tensor $T^{\mu\nu}$ is given in terms of the electromagnetic field tensor $F_{\mu\nu}$ as

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{20}$$

it is readily seen that its trace T vanishes identically. Einstein now entertained the possibility that on a microscopic level all matter might be of electromagnetic origin. In this case, the right hand side of the coordinate condition (19) would vanish and hence coordinates with constant g would be admissible. In this case, Einstein argued, one could take the fully covariant equations

$$R_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{21}$$

which he had already considered earlier, see (8), and reduce them to the field equations (16) by choosing coordinates for which $g \equiv 1$.

The field equations (21) still differ from the final field equations but for the vacuum case, $T_{\mu\nu} = 0$, they are already equivalent. Einstein therefore was able to compute on the basis of (21) the correct unaccounted perihelion advance by looking at the field of a point mass in second approximation. The calculation produced the correct value of $43''$ per century without any arbitrary or ad hoc assumptions. In the computation Einstein could take advantage of his having calculated the advance before for the *Entwurf* theory. The new field equations, in fact, only involved a modification of his earlier calculations (Earman and Janssen 1993). Einstein published these results in his third November communication (Einstein 1915c).

With the success of the perihelion calculation, the return to general covariance was definite. The final step (Einstein 1915d) was to add a trace term to the matter tensor to obtain field equations of the form

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{22}$$

With the trace term added, the postulate of energy-momentum conservation no longer produced a coordinate restriction since it was now automatically satisfied by (22).

Equations (22) are the final field equations of the generally relativistic theory of gravitation, as we know them today. They are frequently referred to as the 'Einstein equations' of general relativity.

With the exception of the first November communication, where he had given the Lagrangian (18) for the field equations (16), Einstein had not discussed the subsequent field equations in a variational approach. The closure of providing a variational formulation was contributed by Hilbert in his own approach to a generally covariant theory of gravitation and electromagnetism. Since he was being kept informed by Einstein about the latter's progress, he rushed ahead and presented an account of his own version to the Göttingen Academy for publication in its *Nachrichten* on November 20. Page proofs of Hilbert's original paper show that the version submitted for publication on November 20 still differed from the version that was eventually published. But it did already suggest to base the theory on a variational principle and emphasized that the Lagrangian must be a scalar function for general coordinate transformations.

In the printed version of Hilbert's paper, the Riemann curvature scalar R is taken to be the gravitational part of the Lagrangian and it is stated, albeit not derived by explicit calculation, that a variation of the action

$$\mathcal{A} = \int \sqrt{-g} \left(R - \kappa L^{(\mathrm{mat})} \right) d^4\tau, \tag{23}$$

with respect to the metric tensor components $g^{\mu\nu}$ would produce the gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa \frac{1}{\sqrt{-g}} \frac{\delta L^{(\mathrm{mat})}}{\delta g^{\mu\nu}}, \tag{24}$$

which is an equivalent version of Einstein's field equation (22). (24) may be transformed to (22) by looking at the trace of (22) and substituting $R = -\kappa T$ into (24). The equivalence then follows from the non-trivial identification of

$$\frac{1}{\sqrt{-g}} \frac{\delta L^{(\mathrm{mat})}}{\delta g^{\mu\nu}} = T_{\mu\nu}. \tag{25}$$

In the latter step, Hilbert and Einstein differed considerably since Hilbert axiomatically took $L^{(\mathrm{mat})}$ to be a function exclusively of the electromagnetic potential A_μ, the electromagnetic field $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, and the metric tensor components $g^{\mu\nu}$,

$$L^{(\mathrm{mat})} = L^{(\mathrm{mat})}(A_\mu, F_{\mu\nu}, g_{\mu\nu}), \tag{26}$$

in accordance with his electromagnetic world view. Einstein, however, had entertained the hypothesis of an electromagnetic origin of matter only for a few days. With his fourth November communication at the latest, Einstein had given up that hypothesis again and was allowing for an unspecified $T_{\mu\nu}$ in his final version of the theory.

9. The 1916 review paper

Ever since Levi-Civita had found a gap in Einstein's covariance proof of the *Entwurf* equations, Einstein had meant to update or rewrite his 1914 Academy article on the general theory of relativity. With the return to general covariance, the success of explaining the perihelion advance of Mercury, and the new field equations (22) of the fourth November communication, Einstein decided to write an altogether new account of the general theory of relativity.

The new review was received by the *Annalen der Physik* on 20 March 1916, some four months after the last November paper. Its structure is not much different from the earlier 1914 Academy article. It is again divided into five sections:

A. Fundamental Considerations on the Postulate of Relativity,

B. Mathematical Aids to the Formulation of Generally Covariant Equations,

C. Theory of the Gravitational Field,

D. Material Phenomena,

E. [Newtonian Limit and Observable Consequences].

In an introductory paragraph Einstein called the theory to be expounded in the review 'conceivably the farthest-reaching generalization' of the special theory of relativity. While the latter is assumed to be known to the reader, he sets out to develop especially all the necessary mathematical tools

> —and I tried to do it in as simple and transparent a manner as possible, so that a special study of the mathematical literature is not required for the understanding of the present paper. (Einstein 1916, 769)

Nevertheless, in this first paragraph Einstein did mention Minkowski's formal equivalence of the spatial and time coordinates, the investigations on non-Euclidean manifolds by Gauss, Riemann, and Christoffel, and the absolute differential calculus of Ricci and Levi-Civita. Echoing a theme of Felix Klein's but also of later commentators, he wrote that especially the absolute differential calculus had provided mathematical means which simply had to be taken up—as if he had not struggled hard for years to apply them in a physically meaningful way. He also acknowledged Grossmann's help again in studying the mathematical literature and in searching for the gravitational field equations.

The first section then introduces the postulate of general covariance, arguing to a large extent from purely epistemological considerations. Einstein denounces the existence of an absolute space by considering two massive bodies far away both from other masses and from each other and in relative rotation along their line of connection. If one body were observed to be of spherical shape and the other to be an ellipsoid, Newtonian mechanics would have to attribute the cause for the different shapes in a rotation relative to absolute space. But this is unsatisfactory because a causal agent is introduced which itself can never be an object of causal effect nor of observation. Hence, one is forced to attribute the cause for this change of shape to the distant masses of the fixed stars, an argument that follows Mach's critique of classical mechanics.

The second argument is the equivalence hypothesis based on Galileo's empirical law of free fall. Next, Einstein discusses the rotating disk to argue for the fact that in general relativity coordinates no longer have an immediate metric meaning. A fourth argument in this section was new and replaced the earlier hole consideration. The hole argument had supposedly proven that no generally covariant field equations could be given a physical meaning in accordance with our notions of causality and the demand that the field equations are determined uniquely by the energy-matter distribution. Einstein did not explicitly retract the argument but gave a new consideration, known as the point coincidence argument. He argued that what we observe in physical experiments are always only spatio-temporal coincidences. If all physical processes would consist in the motion of material points, we could only observe those events where two or more of their worldlines coincide. Then the coordinates of the four-dimensional space-time manifold are merely labels for those coincidences, and no coordinate system must be preferred over any other. The implicit objection to the hole argument that invalidates its conclusion is that the different metric fields $G(x)$ and $G'(x)$ obtained by dragging the metric tensor over the hole, do not, in fact, represent different physical situations since they agree on all point coincidences.

In the second, mathematical section, Einstein summarily develops the elements of tensor algebra and tensor calculus. He introduces contravariant and covariant vectors and general tensors which are defined by the transformation laws of their components. He introduces the algebraic operations of external multiplication and contraction, and of raising and lowering of indices. Among the properties of the metric tensors, he discusses the invariance of the volume element $\sqrt{-g}d^4x$. He repeats the derivation of the geodesic equation, introduces Christoffel's symbols and discusses covariant differentiation by considering invariance along the geodesic line. He mentions the fact that the covariant derivative of the metric vanishes and derives a number of explicit formulas for the differentiation of contravariant, covariant and mixed tensors. The last paragraph introduces the Riemann-Christoffel curvature tensor and discusses its splitting into two parts, as in (13). Perhaps the most noteworthy point of the section, compared to earlier expositions of the mathematical foundations of general relativity, is what came to be called the 'Einstein summation convention'. It is in this section that Einstein for the first time in print introduced the convention that in any tensor expression a summation over two repeated indices is implied with out writing down the summation sign.

The third section derives the gravitational field equations. They are given here as

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{27}$$

$$\sqrt{-g} = 1. \tag{28}$$

Somewhat surprisingly, from a modern point of view, Einstein did not give the field equations in a generally covariant form. Instead he fixed the coordinates by condition (28) in all equations that he gave in the section. He emphasized, though, that this is a mere specification of the coordinates introduced for convenience. The introduction of the field equations, in fact, proceeded by arguing that the vanishing of the Ricci tensor $R_{\mu\nu}$ is the unique equation that determines the metric field in the absence of masses if we demand that the expression depends only on $g_{\mu\nu}$ and its first and second derivatives and depends on the latter only linearly. The possibility of adding a term proportional to $g_{im}R$, equivalent in the vacuum case, (but not of adding a cosmological term proportional to g_{im}) is mentioned in a footnote.

The Lagrangian for the variational form of the field equations in vacuum is given as

$$L = g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}, \tag{29}$$

together with the explicit stipulation of condition (28). The introduction of the matter term proceeds by defining the stress-energy complex of the gravitational field as

$$\kappa t^{\alpha}_{\sigma} = \frac{1}{2}\delta^{\alpha}_{\sigma}g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\sigma}, \tag{30}$$

an expression which is not a tensor under general coordinate transformation in accordance with the fact that the field energy associated with the gravito-inertial field is not a localizable quantity. Using t^α_σ, Einstein rewrote the field equation (27) as

$$\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right)_{,\alpha} = -\kappa\left(t^\sigma_\mu - \frac{1}{2}\delta^\sigma_\mu t\right), \tag{31}$$

and demanded that the non-gravitational energy-momentum tensor T^σ_μ enters in the equation on the same footing as t^σ_μ. The latter requirement is equivalent to demanding that a divergence equation,

$$\left(t^\sigma_\mu + T^\sigma_\mu\right)_{,\sigma} = 0, \tag{32}$$

holds for the total energy of the system.

While the derivation of the field equations differs considerably from earlier accounts, the fourth and fifth sections take up material from earlier expositions. In these sections, Einstein discussed Euler's hydrodynamic equation with an energy-momentum tensor

$$T^{\alpha\beta} = -g^{\alpha\beta}p + \rho u^\alpha u^\beta, u^\alpha = dx^\alpha/ds, \tag{33}$$

for non-dissipative, adiabatic liquids, characterized by the two scalars of pressure p and density ρ. Electrodynamics is governed by Maxwell's equations in generally covariant form, and, in the last section, Einstein discussed the Newtonian approximation of weak fields, Minkowski flat boundary conditions, and slow motion of the particles. In the consideration of the Newtonian limit the constant κ may be related to the gravitational constant G by comparison with Poisson's equation as $\kappa = 8\pi G/c^2$. Einstein explains a subtlety of the Newtonian limit that had played a role in his earlier dismissal of generally covariant equations. In first Newtonian approximation only the g_{44} components enter into the equations of motion, even though the postulate of $\sqrt{-g} = 1$ demands that the other diagonal components are non-trivial of the same order. The first-order diagonal components, however, do enter into the geodesic equation for a light ray passing in a centrally symmetric gravitational field. For this reason, the predicted expression for the light bending of a light ray grazing the edge of the sun, came out with a factor of 2, compared to earlier considerations that were based on the equivalence hypothesis alone. The slowing of clocks in a gravitational field and the gravitational red shift of spectral lines is discussed explicitly but the calculation of the perihelion shift for Mercury obtained in second approximation of a spherically symmetric field is only mentioned with reference to the pertinent November communication.

10. Early reception of the final version of general relativity

The first exact solution to the field equations—and to date the most important one—was found almost simultaneously with Einstein's 1916 review by the astronomer Karl Schwarzschild (1873–1916). He computed the static, spherically symmetric field outside a spherically symmetric mass distribution of total mass m. His solution allowed to compute the light bending of light rays and the planetary perihelion motion without approximation. The solution is regular everywhere except at the origin but at a radius $r_S = 2Gm/c^2$, now called the Schwarzschild radius, the time coordinate changes its sign relative to the spatial coordinates. This coordinate singularity is responsible for what came to be known as the black hole horizon and its interpretation presented a major difficulty for many years.

While more exact solutions were found over the following years, approximation schemes played an equally important role for an interpretation of the theory. An approximate solution was discussed by Einstein in the summer of 1916 in a first paper on gravitational waves. The existence of gravitational waves was expected in a field theory of gravitation by analogy to the electromagnetic case. Einstein's first paper on this topic was marred by a mistake which made him conclude that waves should exist that do not transport energy. The error was corrected in a second paper of 1918. Until now, the topic of gravitational waves is an active field of research and their existence has been shown indirectly only in 1974 through the energy loss of binary pulsars (Nobel prize 1993). Experimental efforts to observe gravitational waves directly are still underway.

The question of energy transport in gravitational waves is connected to the question of identifying an expression for the gravitational field energy and a corresponding conservation law. The question was debated in the years 1916–1919 by a number of mathematicians, most importantly by Felix Klein. The final solution came with Noether's theorems on the connection of conservation laws and symmetries of the variational formulation. These theorems were anticipated for a special case in Hilbert's 1915 paper and published in its general form in 1918 by Emmy Noether (1882–1935).

Einstein tried to encourage experimental efforts aimed at testing the two main predictions of the theory. A confirmation of the gravitational red shift was difficult to determine due to the many competing effects that result in a shifting or broadening of solar or stellar spectral lines. An unequivocal confirmation of the gravitational red shift only came in 1960 in a controlled terrestrial experiment making use of the Mössbauer effect.

But the results of a British expedition led by Arthur Eddington (1882–1944) to test the predicted gravitational light bending during a solar eclipse on 29

May 1919 in Sobral, Brazil, and on the island of Principe in the gulf of Guinea, were established and reached Europe later in the fall of that year. The results confirmed Einstein's prediction, and within weeks, Einstein turned into a world celebrity and the theory of relativity into a household term.

A popular, non-technical account of both the special and general theories of relativity that Einstein had written in 1917 (Einstein 1917) became a best-seller. A fourth edition in 1919 was reprinted in a fifth through tenth edition in 1920 and saw a fourteenth edition in 1922. It was also translated into many languages. The increased interest in Einstein's theory is also witnessed by an uncountable number of more or less popular accounts and other books and articles dealing with relativity. A bibliography of relativity from 1924 lists close to 4000 entries (Lecat 1924).

The consequences of both special and general relativity began to be discussed in many circles. Early interpretations of general relativity from a philosophical point of view had been published by Moritz Schlick (1882–1936) and Hans Reichenbach (1891–1953). In the early 1920's philosophical interpretations of relativity came to abound, the analysis in (Hentschel 1990) carries a bibliography of over 3000 items. The public interest in Einstein's new theory was not always untainted by political partisanry. Antisemitic attacks against Einstein focussed not only on Einstein's person or on his political and pacifist stance but targeted his theory as well. As early as 1920, antisemitically motivated objections against the theories of relativity were expressed in a public meeting at the Berlin philharmonic in summer 1920 and again at the first post-war meeting of the Society of German Scientists and Physicians in Bad Nauheim in September 1920. On the other hand, Einstein began to be recognized worldwide as a leading physicist. He received international invitations and honors, and began to travel extensively giving talks about his theory at a time when post-war German science was still boycotted by many scholars and scientific institutions.

11. Going on and beyond general relativity

For Einstein, the victory of the breakthrough to general covariance in November 1915 was not to be regarded as establishing a final theory that would not be subject to further revisions. Already in 1917, he modified the gravitational field equations by adding a term proportional to $\lambda g_{\mu\nu}$ to (22). The modification was motivated in the context of a cosmological consideration. Einstein wanted to avoid the stipulation of boundary conditions at infinity in order not to have to account for inertial effects that might not have been caused by masses, in accordance with what he called Mach's principle. He suggested to consider the cosmological model of a spatially closed and static universe but had to modify the field equations by introducing the cosmological constant λ in order to allow for the possibility of such a solution. An alternate vacuum solution to the modified field equations advanced by Willem de Sitter (1872–1934) soon showed, however, that the new field equations did not automatically satisfy Mach's principle as had been Einstein's hope.

In 1919, Einstein entertained the possibility of a gravitational field equation where the trace term in (22) would be added with a factor of 1/4 instead of 1/2. The modification was motivated by considerations concerning the constitution of matter and implies that it is no longer the covariant divergence of $T_{\mu\nu}$ that is automatically vanishing but rather its trace. Other modifications of the field equations or generalizations of the underlying Riemannian geometry were investigated by Einstein and others in the following decades in attempts to find a geometrized unification of the gravitational and electromagnetic fields.

In fact, a geometric interpretation of the general theory of relativity, if considered at all, originally pertained only to the geodesic equation. Until 1916, the Riemann and Ricci tensors were only interpreted as algebraic invariants. A geometric interpretation in terms of parallel transport of tangent vectors was elaborated in the following years mainly through the work of Tullio Levi-Civita and Hermann Weyl (1885–1955).

In the course of elaborating the geometric meaning of general relativity, it was Hermann Weyl, who took the first steps to go beyond a purely (semi-)Riemannian framework for general relativity and, at the same time, first proposed a truly geometrized unification of the gravitational and electromagnetic fields. First published in 1918, it was later incorporated into the third edition of his widely read exposition of general relativity (Weyl 1918), (Scholz 2001). In accordance with more general philosophical concerns about the foundations of mathematics, Weyl's point of departure was the observation that in Riemannian geometry, no integrable, or path-independent comparison of vector directions at different points of the manifold is possible, whereas the length of a vector remains unaffected during parallel transport. In order to realize a true 'infinitesimal geometry' (*Nahegeometrie*), Weyl in 1918 introduced an additional geometric structure, a length connection, i.e. a linear differential form $d\varphi = \varphi_i dx^i$ that governed the transport of vector lengths l by the definition $\delta l \equiv (\partial l/\partial x^i)dx^i + l\varphi_i dx^i \equiv 0$. At the same time, the Riemannian metric $g_{\mu\nu}$ had to be replaced by the class of conformally equivalent metrics $[g]$ where two representatives of a class are connected through $\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$ with a scalar function λ. For consistency, the length connection φ has to be transformed, too, as

$\tilde{\varphi}_i dx^i = \varphi_i dx^i - d \log \lambda$. For these transformations, Weyl introduced the term 'gauge transformations.'

The (semi-)Riemannian manifold with metric tensor field $g_{\mu\nu}$ was hence generalized to a manifold with conformally equivalent classes $[g]$, $[\varphi]$ of (semi-)Riemannian metrics and length connections. The geometric meaning of this generalization was realized by investigating the affine connection, governing the parallel transport of vectors. It turned out that the curvature associated with the length connection, i.e. the exterior derivative of $f = d\varphi$, in coordinates, $f_{ij} = \varphi_{i,j} - \varphi_{j,i}$, could be interpreted as the representation of the electromagnetic field tensor (Scholz 2001, esp. pp. 63–69).

Einstein's reaction to Weyl's theory was highly ambivalent. Fascinated by the mathematical analysis, he quickly pointed out that the theory was inacceptable from a physics point of view since it implied, e.g., that the wavelength of light emitted by radiating atoms should depend on the prehistory of the atom, contrary to experience. Despite this argument, Weyl's theory proved extremely influential as the first (more or less) successful attempt to achieve a geometric unification of the gravitational and electromagnetic fields. During the twenties, many attempts were tried to achieve a unification of gravitation and electromagnetism by generalizing Riemann geometry. These investigations both stimulated and profited from parallel developments in differential geometry.

With the advent of quantum mechanics in 1926, the discovery of the weak and strong interactions and the proliferation of elementary particles in nuclear and subnuclear physics, the parameters for a unification program changed drastically. Many aspects of the original unified field theory program have consequently fallen into oblivion but the history of modern differential geometry can hardly be understood without taking into account this context of searching for generalizations of Riemannian geometry.

In essence, Einstein's general theory of relativity of 1916 remains today's accepted theory of the gravitational field, and notwithstanding the expectation that a generally relativistic theory of gravitation should also be quantized—an unsolved problem until today—, classical general relativity, in the sense of an exploration of the solutions and implicit consequences of its gravitational field equations, has been an active field of research ever since.

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