CHAPTER 10
Advance of Mercury's Perihelion
A Project

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This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right. "For a few days, I was beside myself with joyous excitement." Later, he told Fokker that his discovery had given him palpitations of the heart. What he told de Haas is even more profoundly significant: when he saw that his calculations agreed with the unexplained astronomical observations, he had the feeling that something actually snapped in him.

—Abraham Pais

■ JOYOUS EXCITEMENT

Tiny effect; large significance.

What discovery sent Einstein into "joyous excitement" in November of 1914? It was his calculation showing that his brand new (actually, not quite completed) theory of general relativity gave the correct value for one detail of the orbit of the planet Mercury that had previously been unexplained, named the **precession of Mercury's perihelion**, which we now describe.

Newton: Sun-Mercury alone: perihelion fixed.

Mercury circulates around the Sun in a not-quite-circular orbit; like the other planets of the solar system, it oscillates in and out radially while it circles tangentially. The result is an elliptical orbit. Newton tells us that if we consider only the interaction between Mercury and the Sun, then the time for one 360-degree trip around the Sun is *exactly* the same as one in-and-out radial oscillation. Therefore the orbital point closest to the Sun, the so-called **perihelion,** stays in the same place; the elliptical orbit does not shift around with each revolution—according to Newton. In this project you will begin by verifying this nonrelativistic result for the Sun-Mercury system alone.

However, observation shows that Mercury's orbit does, in fact, change. The innermost point, the perihelion, moves around the Sun *slowly*; it *advances* with each orbit (Figure 1). The long (major) axis of the ellipse rotates. We call this rotation of the axis the **precession of the perihelion**. The perihelion of

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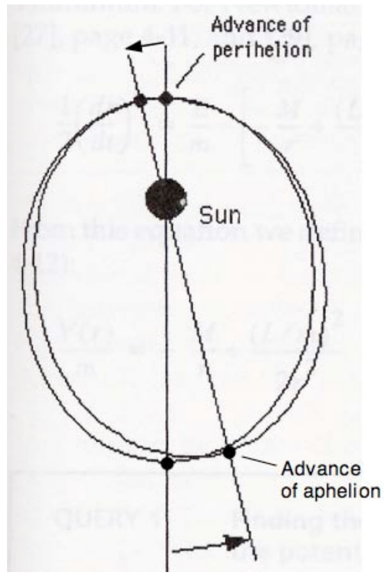

FIGURE 1 Exaggerated view of the advance during one century of Mercury's perihelion (and aphelion). The figure shows two elliptical orbits. One of these orbits is the one that Mercury traces over and over again almost exactly in the year, say 1900. The other elliptical orbit is the one that Mercury traces over and over again almost exactly in the year, say 2000. The two are shifted with respect to one another, a changed called *the advance (or precession) of Mercury's perihelion.* The actual rotation (due to general relativity) is about 43 seconds of arc, which is $43/3600 \approx 0.0119$ degree, corresponding to less than the thickness of a line in this figure.

Newton: Influence of other planets, predicts most of the perihelion advance . . .

Mercury actually precesses at the tiny rate of 574 seconds of arc (0.159 degree) *per century.* (One degree equals 3600 seconds of arc.) The perihelion moves forward in the direction of rotation of Mercury; it advances. The **aphelion** is the point of the orbit farthest from the Sun; it advances at exactly the same angular rate as the perihelion (Figure 1).

Newtonian mechanics accounts for 531 seconds of arc of this advance by computing the perturbing influence of the other planets. But a stubborn 43 seconds of arc (0.0119 degree) per century, called a **residual**, remains after all these effects are accounted for. This residual (though not its modern value) was computed from observations by Urbain Le Verrier as early as 1859 and more accurately later by Simon Newcomb (Box 1). Le Verrier attributed the residual in Mercury's orbit to the presence of an unknown inner planet, tentatively named Vulcan. Of course there is no planet Vulcan.

. . . but not all of it.

Einstein's general relativity predicts extra observed precession.

Newtonian mechanics says that there should be *no residual* advance of the perihelion of Mercury's orbit and so cannot account for the 43 seconds of arc per century which, though tiny, is nevertheless too large to be ignored or blamed on observational error. But Einstein's general relativity hit 43 seconds of arc on the button. Result: joyous excitement!

BOX 1. Simon Newcomb

FIGURE 2 Simon Newcomb
Born 12 March 1835, Wallace, Nova Scotia.
Died 11 July 1909, Washington, D.C.
(Photo courtesy of Yerkes Observatory)

From 1901 until 1959 and even later, the tables of locations of the planets (so-called **ephemerides**) used by most astronomers were those compiled by Simon Newcomb and his collaborator George W. Hill. By the age of five Newcomb was spending several hours a day making calculations, and before the age of seven was extracting cube roots by hand. He had little formal education but avidly explored many technical fields in the libraries of Washington, D. C. He discovered the *American Ephemeris and Nautical Almanac,* of which he said, "Its preparation seemed to me to embody the highest intellectual power to which man had ever attained."

Newcomb became a "computer" (a person who computes) in the American Nautical Almanac Office and, by stages, rose to become its head. The greater part of the rest of his life was spent calculating the motions of bodies in the solar system from the best existing data. Newcomb collaborated with Q. M. W. Downing to inaugurate a worldwide system of astronomical constants, which was adopted by many countries in 1896 and officially by all countries in 1950.

The advance of the perihelion of Mercury computed by Einstein in 1914 would have been compared to entries in the tables of Simon Newcomb and his collaborator.

Method: Compare in-and-out time with round-and-round time for Mercury.

Preview: In this project we develop an approximation to demonstrate Newton's no-precession conclusion, then carry out an approximate general-relativistic calculation that predicts precession. Both approximations assume that Mercury is in a near-circular orbit. From this assumption we calculate the time for one orbit. The approximation also describes the small inward and outward radial motion of Mercury as if it were a harmonic oscillator moving back and forth radially about the minimum in a potential well centered at the radius of the circle (Figure 3). We calculate the time for one round-trip radial oscillation. The orbital and radial oscillation times are equal, according to Newton, if one considers only the Mercury-Sun interaction. In that case Mercury goes around once in the same time that it oscillates radially inward and back out again. The result is an elliptical orbit that closes on itself, so that in the absence of other planets Mercury repeats its elliptical path forever. In contrast, our general relativity approximation shows that these two times—the angular round-and-round time and the radial in-and-out time—are *not quite* equal. The radial oscillation takes place more slowly, so that by the time Mercury returns to its maximum orbital radius the circular motion has carried it farther around the Sun than it was at the preceding maximum radius. From this difference we reckon the approximate rate of advance of Mercury's perihelion around the Sun. Now for the details.

2 ■ SIMPLE HARMONIC OSCILLATOR

Assume radial oscillation is sinusoidal.

Why should the satellite oscillate in and out radially? Look at the effective potential for Newtonian motion, the heavy line in Figure 3. This heavy line has a minimum, the location at which a particle can ride around at constant r, executing a circular orbit. But with a slightly higher energy, it can also oscillate radially in and out, as shown by the two-headed arrow.

How long will it take for one in-and-out oscillation? That depends on the shape of the effective potential curve near the minimum shown in Figure 3. If the amplitude of the oscillation is small, then the effective part of the curve is very close to this minimum, and we can use a well-known mathematical theorem: If a continuous, smooth curve has a local minimum, then near that minimum the curve can be approximated by a parabola with its vertex at the minimum point. Figure 3 shows such a parabola (thin curve) superimposed on the (heavy) effective potential curve. From the diagram it is apparent that the parabola is a good approximation of the potential near that local minimum. In fact Mercury's orbit swings from a minimum radius (the perihelion) of 46.04 million kilometers to a maximum radius (the aphelion) of 69.86 million kilometers. The difference in radius is not a small fraction of the minimum radius of the orbit; nevertheless our approximate analysis yields a decent result compared to the observed precession.

In-and-out motion in parabolic potential . . .

. . . predicts simple harmonic motion.

From introductory physics we know how a particle moves in a parabolic potential. The motion is called **simple harmonic oscillation**, described by the following expression:

$$x = \mathrm{A}\sin\omega t \tag{1}$$

Here A is the amplitude of the oscillation and ω (Greek lower case omega) tells us how rapidly the oscillation occurs in radians per unit of time. The potential energy per unit mass, V/m, of a particle oscillating in a parabolic potential follows the formula

$$\frac{V}{m} = \frac{1}{2}\omega^2 x^2 \tag{2}$$

From (2) we find an expression for ω by taking the second derivative of both sides with respect to the displacement x:

$$\frac{d^2\left(V/m\right)}{dx^2} = \omega^2 \tag{3}$$

Taken together, (2) and (3) show that for the simple harmonic oscillator the radial oscillation frequency ω does not depend on the amplitude A. In general, if we have the expression for the potential, we can find the rate ω of harmonic oscillation around a local minimum by taking the second derivative of the curve and evaluating it at that minimum.

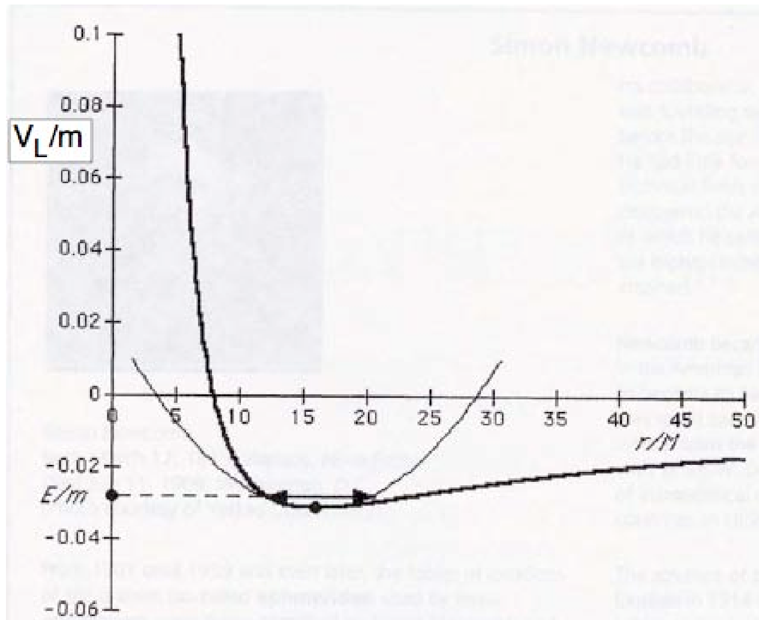

FIGURE 3 Newtonian effective potential (heavy curve), copied from Chapter 4, on which is superimposed the parabolic potential of the simple harmonic oscillator (thin curve). The two curves conform to one another only near the minimum of the effective potential. We use a similar set of curves to approximate the radial oscillation of Mercury in its orbit as an harmonic oscillation of small amplitude.

3 ■ RADIAL HARMONIC OSCILLATION OF MERCURY: NEWTON

Oscillating radially in and out about what center?

The trouble with the in-and-out radial oscillation of Mercury is that it does not take place around $x = 0$ but around the average radius r_0 of its orbit. What is the value of r_0? It is the radius at which the effective potential is minimum. For Newtonian orbits the radial motion is given in Chapter 4:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left[-\frac{M}{r} + \frac{(L/m)^2}{2r^2}\right] = \frac{E}{m} - \frac{V_{\mathrm{L}}(r)}{m} \qquad \text{(Newton)} \qquad (4)$$

This equation defines the effective potential,

$$\frac{V_{\mathrm{L}}(r)}{m} \equiv -\frac{M}{r} + \frac{(L/m)^2}{2r^2} \qquad \text{(Newton)} \qquad (5)$$

QUERY 1. Find the local minimum of the potential
Take the derivative with respect to r of the potential per unit mass, V_{L}/m, given in (5). Set this first derivative aside for use in Query 2. As a separate calculation, equate this derivative to zero in order to

determine the radius r_0 at the local minimum of the effective potential. Use the result to write down an expression for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_0.

QUERY 2. Oscillation rate ω_r for radial motion
We want to use (3) to find the rate of radial oscillation. Accordingly, continue by taking a second derivative of V_L/m in (5) with respect to r. Set $r = r_0$ in the resulting expression and substitute your value for $(L/m)^2$ from Query 1. Use (3) to find an expression for the rate ω_r at which Mercury oscillates in and out radially—according to Newton!

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4 ■ ANGULAR VELOCITY OF MERCURY IN ITS ORBIT: NEWTON

Round and round how fast?

Newton: in-and-out time equals round-and-round time.

We want to compare the rate ω_r of in-and-out radial motion of Mercury with its rate ω_ϕ of round-and-round tangential motion. Use the Newtonian definition of angular momentum, with increment dt of Newtonian universal time, similar to equation (10), page 4 of Chapter 4:

$$L/m = r^2\frac{d\phi}{dt} = r^2\omega_\phi \qquad \text{(Newton)} \tag{6}$$

Now find the value for the angular velocity $\omega_\phi = d\phi/dt$ of Mercury along its almost circular orbit.

QUERY 3. Angular velocity of Mercury in orbit.
Into (6) substitute your value for L/m from Query 1 and set $r = r_0$. Find an expression for ω_ϕ in terms of M and r_0.

QUERY 4. Comparing radial oscillation rate with orbital angular velocity.
Compare your value of angular velocity ω_ϕ from Query 3 with your value for radial oscillation rate ω_r from Query 2. Will there be an advance of the perihelion of Mercury's orbit around the Sun (when only the Sun-Mercury interaction is considered)—according to Newton?

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5 ■ EFFECTIVE POTENTIAL: EINSTEIN

Added effective potential term advances perihelion.

Now we repeat the analysis for the general relativistic case, using the Newtonian analysis as our model. Chapter 9, Orbiting, predicts the radial motion of an orbiting satellite. Multiply equations (29) and (30) of that chapter through by 1/2 to obtain an equation similar to (4) above for the Newtonian case:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2}\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right] \tag{7}$$

$$= \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2}\left(\frac{V_{\mathrm{L}}}{m}\right)^2 \qquad \text{(Einstein)}$$

Set up general relativity effective potential.

Equations (4) and (7) are of similar form, and we use this similarity to make a general relativistic harmonic analysis of the radial motion of Mercury in orbit. In this process we adopt the *algebraic manipulations* of the Newtonian analysis of Sections 3 and 4 but apply them to the general relativistic expression (7).

Different times on different clocks do not matter.

Before proceeding, note three characteristics of equation (7). First, the time on the left side of (7) is the proper time τ, the wristwatch time of the planet Mercury, not Newton's universal time t. This different time standard is not necessarily fatal, since we have not yet decided which relativistic time should replace Newton's universal time t. You will show in Section 11 that for Mercury the choice of which time to use (wristwatch time, Schwarzschild map time, or even shell time at the radius of the orbit) makes a negligible difference in our predictions about the rate of advance of the perihelion.

Note second that in equation (7) the relativistic expression $(1/2)(E/m)^2$ stands in the place of the Newtonian expression E/m in (4). However, both are constant quantities, which is all that matters in carrying out the analysis. Evidence that we are on the right track follows from multiplying out the second term of the middle equality in (7). Note that we have assigned the symbol $(1/2)(V_{\mathrm{L}}/m)^2$ to this second term.

$$\frac{1}{2}\left(\frac{V_{\mathrm{L}}}{m}\right)^2 = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right] \qquad \text{(Einstein)} \tag{8}$$

$$= \frac{1}{2} - \frac{M}{r} + \frac{(L/m)^2}{2r^2} - \frac{M\,(L/m)^2}{r^3}$$

Details of relativistic effective potential

The heavy curve in Figure 4 plots this function. The second line in (8) contains the two effective potential terms that made up the Newtonian expression (5). In addition, the first term assures that far from the center of attraction the radial speed in (7) will have the correct value. For example, let the energy equal the rest energy ($E/m = 1$). Then for large r, the radial speed $dr/d\tau$ in (7) goes to zero, as it must in this case. The final term on the right of the second line of (8) describes an attractive potential arising from general relativity. For the Sun-Mercury case, at the radius of Mercury's orbit this term results in the slight precession of Newton's elliptical orbit. In the case of the black hole, quite close to its center the r^3 in the denominator causes this term to overwhelm all other terms in (8), leading to the downward plunge in the effective potential at the left side of Figure 4.

Third about equation (7): The final term $(1/2)(V_{\mathrm{L}}/m)^2$ takes the place of the effective potential V_{L}/m in equation (4) of the Newtonian analysis.

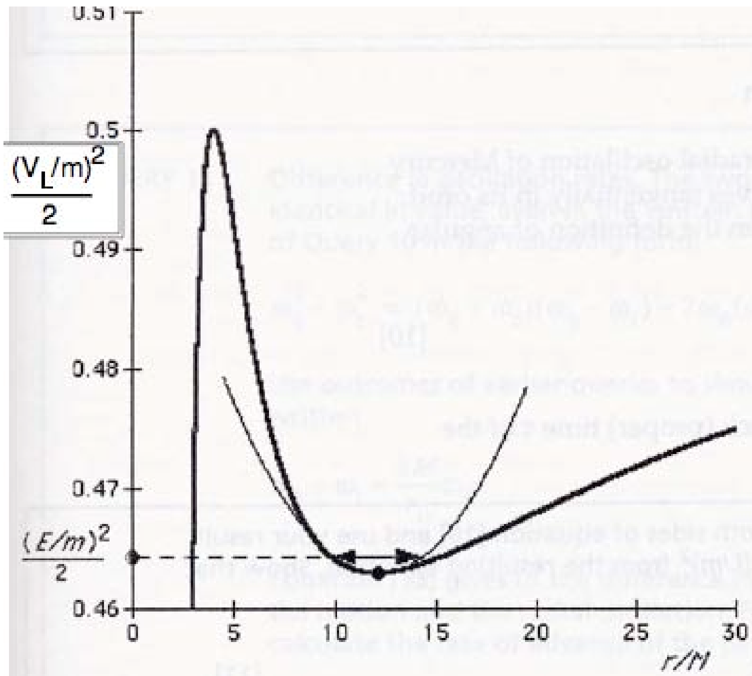

FIGURE 4 General-relativistic effective potential $(1/2)(V_{\rm L}/m)^2$ (heavy curve) and its approximation at the local minimum by a parabola (light curve) in order to analyse the radial excursion (double-headed arrow) of Mercury as simple harmonic motion. The effective potential curve is for a black hole, not for the Sun, whose effective potential near the potential minimum would be indistinguishable from the Newtonian function on the scale of this diagram—even though this minute difference accounts for the tiny precession of Mercury's orbit.

In summary, we can manipulate relativistic expressions (7) and (8) in almost exactly the same way that we manipulated Newtonian expressions (4) and (5) in order to analyze the radial component of Mercury's motion and the small perturbations of the Newtonian elliptical orbit brought about by general relativity.

6 ■ RADIAL HARMONIC OSCILLATION OF MERCURY: EINSTEIN

Einstein tweaks Newton's solution.

Now analyze the radial oscillation of Mercury according to Einstein.

QUERY 5. Finding the local minimum of the effective potential

Take the derivative of the effective potential (8) with respect to r, that is find $d[(1/2)(V_{\rm L}/m)^2]/dr$. Set this first derivative aside for use in Query 6. As a separate calculation, equate this derivative to zero, set $r = r_0$, and solve the resulting equation for the unknown quantity $(L/m)^2$ in terms of the known quantities M and r_0.

QUERY 6. Radial oscillation rate.
We want to use (3) to find the rate of oscillation in the radial direction. Accordingly, continue to the second derivative of $(1/2)(V_{\rm L}/m)^2$ from (8) with respect to r. Set $r = r_0$ in the result and substitute the expression for $(L/m)^2$ from Query 5 to obtain

$$\left[\frac{d^2}{dr^2}\left\{\frac{1}{2}\left(\frac{V_{\rm L}}{m}\right)^2\right\}\right]_{r=r_0} = \omega_{\rm r}^2 = \frac{M\,(r_0 - 6M)}{r_0^3\,(r_0 - 3M)} \tag{9}$$

QUERY 7. Newtonian limit of radial oscillation.
The average radius of Mercury's orbit around the Sun has the value $r_0 = 5.80 \times 10^{10}$ meters. Compare this radius with the value M for the mass of the Sun in geometric units. If one of these can be neglected in (9) compared with the other, demonstrate that the resulting value of $\omega_{\rm r}$ is the same as your Newtonian expression derived in Query 2.

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7 ■ ANGULAR VELOCITY IN ORBIT: EINSTEIN

In-out motion not in step with round-and-round motion.

We want to compare the rate of in-and-out radial oscillation of Mercury with the angular rate at which Mercury moves tangentially in its orbit. The rate of change of azimuth ϕ springs from the definition of angular momentum in Chapter 4:

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \tag{10}$$

Note that the time here, too, is the wristwatch (proper) time τ of the satellite.

QUERY 8. Angular velocity. Square both sides of (10) and use your result from Query 5 to eliminate $(L/m)^2$ from the resulting equation. Show that the result can be written

$$\omega_\phi^2 \equiv \left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r_0^2\,(r_0 - 3M)} \tag{11}$$

According to the relativistic prediction, does the round-and-round tangential motion of Mercury take place in step with the in-and-out radial oscillation, as it does in the Newtonian analysis?

QUERY 9. Newtonian limit of angular velocity.
Make the same kind of approximation as in Query 7 and demonstrate that the resulting value of ω_ϕ is the same as your Newtonian expression derived in Query 3.

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8 ■ PREDICTING ADVANCE OF THE PERIHELION

Simple outcome

Einstein: in-out rate differs from round-round rate.

The advance of the perihelion of Mercury springs from the difference between the frequency at which the planet sweeps around in its orbit and the frequency at which it oscillates in and out radially. In the Newtonian analysis these two frequencies are equal if one considers only the interaction between Mercury and the Sun. But Einstein's theory shows that these two frequencies are *not quite* equal, so Mercury reaches its maximum (and also its minimum) radius at a slightly different angular position in each successive orbit. This results in the advance of the perihelion.

The time to make a complete in-and-back-out radial oscillation is

$$T_{\mathrm{r}} \equiv \frac{2\pi}{\omega_{\mathrm{r}}} \tag{12}$$

In this time Mercury goes around the Sun through an angle, in radians:

$$\omega_\phi T_{\mathrm{r}} = \frac{2\pi\omega_\phi}{\omega_{\mathrm{r}}} = (\text{Mercury rotation angle in time } T_{\mathrm{r}}) \tag{13}$$

which exceeds one complete revolution by (radians):

$$\omega_\phi T_{\mathrm{r}} - 2\pi = T_{\mathrm{r}}\left(\omega_\phi - \omega_{\mathrm{r}}\right) = (\text{excess angle per revolution}) \tag{14}$$

QUERY 10. Difference in oscillation rates.
The two angular rates ω_ϕ and ω_ϕ are *almost* identical in value, even in the Einstein analysis. Therefore write the result of equations (9) and (11) in the following form:

$$\omega_\phi^2 - \omega_{\mathrm{r}}^2 = \left(\omega_\phi + \omega_{\mathrm{r}}\right)\left(\omega_\phi - \omega_{\mathrm{r}}\right) \approx 2\omega_\phi\left(\omega_\phi - \omega_{\mathrm{r}}\right) \tag{15}$$

Complete the derivation to show that in this approximation

$$\boxed{\omega_\phi - \omega_{\mathrm{r}} \approx \frac{3M}{r_0}\omega_\phi} \tag{16}$$

Equation (16) gives us the difference in angular rate between the tangential motion and the radial oscillation. From this rate difference we can calculate the rate of advance of the perihelion of Mercury.

UNITS: The symbol ω in (16) expresses rotation rate in radians per unit of time. What unit of time? It does not matter, as long as the unit of time is the *same* on both sides of the equation. In the following Queries you use (16) to calculate the precession rate of Mercury in radians per Earth century, then convert the result first to degrees per century and finally to seconds of arc per century.

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9 ■ COMPARISON WITH OBSERVATION

You check out Einstein.

Now you can compare our approximate relativistic prediction with observation.

QUERY 11. Mercury's orbital period.
The period of Mercury's orbit is 7.602×10^6 seconds and that of Earth is 3.157×10^7seconds. What is the value of Mercury's period in Earth-years?

QUERY 12. Mercury's revolution in one century.
How many revolutions around the Sun does Mercury make in one century (in 100 Earth-years)?

QUERY 13. Correction factor
The mass M of the Sun is 1.477×10^3 meters and the radius r_0 of Mercury's orbit is 5.80×10^{10} meters. Calculate the value of the correction factor $3M/r_0$ in (16).

QUERY 14. Advance angle in degrees per century.
From equation (16) derive a numerical prediction of the advance of the perihelion of Mercury's orbit in degrees per century (assuming only Mercury and the Sun are interacting).

QUERY 15. Advance angle in seconds of arc per century.
There are 60 minutes of arc per degree and 60 seconds of arc per minute of arc. Multiply your result from Query 14 by $60 \times 60 = 3600$ to obtain your prediction of the advance of the perihelion of Mercury's orbit in seconds of arc per century.

Observation and careful calculation match for Mercury.

A more careful relativistic analysis predicts a value of 42.980 seconds of arc (0.0119 degrees) per century (see Table 4). The observed rate of advance of the perihelion is in perfect agreement with this value: 42.98 ± 0.1 seconds of arc per century. (See references.) How close was your prediction?

Neutron star binary: faster precession.

For comparison, the precession rate exceeds 4 degrees per year for the binary neutron star systems called B1913+16 and J0737-3039. In these cases, each neutron star in the binary system orbits the center of mass in a nearly elliptical orbit whose point of closest approach (technically called **periastron**, not perihelion for a center of attraction other than the Sun) shifts in angular position due to the same effects as those present in the Mercury-Sun system, only thousands of times stronger. See the references.

10 ■ ADVANCE OF THE PERIHELIA OF THE INNER PLANETS

Help from a supercomputer.

All planet orbits precess.

Do the *perihelia* (plural of *perihelion*) of other planets in the solar system also advance as described by general relativity? Yes, but these planets are farther from the Sun, so the magnitude of the predicted advance is less than that for Mercury. In this section we compare our estimated advance of the perihelia of the inner planets Mercury, Venus, Earth, and Mars with results of an accurate calculation.

TABLE 4 Advance of the perihelia of the inner planets

Planet	Advance of perihelion in seconds of arc per century (JPL calculation)	Radius of orbit in AU*	Period of orbit in years
Mercury	42.980 ± 0.001	0.38710	0.24085
Venus	8.618 ± 0.041	0.72333	0.61521
Earth	3.846 ± 0.012	1.00000	1.00000
Mars	1.351 ± 0.001	1.52368	1.88089

*Astronomical Unit (AU): average radius of Earth's orbit; inside front cover.

The Jet Propulsion Laboratory (JPL) in Pasadena, California, supports an active effort to improve our knowledge of the positions and velocities of the major bodies in the solar system. For the major planets and the moon, JPL maintains a database and set of computer programs known as the Solar System Data Processing System (SSDPS). The input database contains the observational data measurements for current locations of the planets. Working together, more than 100 interrelated computer programs use these data and the relativistic laws of motion to compute locations of planets at times in the past and future. The equations of motion take into account not only the gravitational interaction between each planet and the Sun but also interactions among all planets, Earth's moon, and 300 of the most massive asteroids, as well as interactions between Earth and Moon due to nonsphericity and tidal effects.

JPL multi-program computation.

To help us with our project on perihelion advance, Myles Standish, Principal Member of the Technical Staff at JPL, kindly used the numerical integration program of the SSDPS to calculate orbits of the four inner planets over four centuries, from A.D. 1800 to A.D. 2200. In an overnight run he carried out this calculation twice, once with the full program including relativistic effects and a second time "with relativity turned off." Standish "turned off relativity" by setting the speed of light to 10^{10} times its measured value, effectively making light speed infinite. The coefficient of dt^2 in the Schwarzschild metric is written in conventional units as $1 - 2GM_{\text{conv}}/(rc^2)$. The value of this expression approaches unity for a large value of c.

For each of the two runs, the perihelia of the four inner planets were computed for a series of points in time covering the four centuries. The results from the nonrelativistic run were subtracted from those of the relativistic run, revealing advances of the perihelia per century accounted for only by general relativity. The second column of Table 4 shows the results, together with the estimated computational error.

QUERY 16. Perihelia advance of the inner planets.
Compare the JPL-computed advances of the perihelia of Venus, Earth, and Mars with results of the approximate formula developed in this project.

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11 ■ CHECKING THE STANDARD OF TIME

Whose clock?

We have been casual about whose time tracks the advance of the perihelion of Mercury and other planets. Does this invalidate our approximations?

QUERY 17. Difference between shell time and Mercury wristwatch time. Use special relativity to find the fractional difference between planet Mercury's wristwatch time increment $\Delta\tau$ and the time increment Δt_{shell} read on shell clocks at the same average radius r_0 at which Mercury moves in its orbit at the average velocity 4.8×10^4 meters/second. By what fraction does a change of time from $\Delta\tau$ to Δt_{shell} change the total angle covered in the orbital motion of Mercury in one century? Therefore by what fraction does it change the predicted angle of advance of the perihelion in that century?

QUERY 18. Difference between shell time and Schwarzschild map time. Find the fractional difference between shell time increment Δt_{shell} at radius r_0 and Schwarzschild map time increment Δt for r_0 equal to the radius of the orbit of Mercury. By what fraction does a change of time from Δt_{shell} to t change the predicted angle of advance of the perihelion in that century?

QUERY 19. Does time standard matter? From your results in Queries 17 and 18, say whether or not the choice of a time standard—wristwatch time of Mercury, shell time, or map time—makes a detectable difference in the numerical prediction of the advance of the perihelion of Mercury in one century. Would your answer differ if the time were measured with clocks on Earth's surface?

APPRECIATING SOME DEEP INSIGHTS FROM MORE THAN 300 YEARS AGO

> *Newton himself was better aware of the weaknesses inherent in his intellectual edifice than the generations that followed him. This fact has always roused my admiration.*
>
> —Albert Einstein

We agree with Einstein. In the following quote from the end of his great work *Principia*, Isaac Newton tells us what he knows and what he does not know about gravity. The scope of what he says—and the integrity of what he refuses to say—are breathtaking.

"I do not 'feign' hypotheses."

> *Thus far I have explained the phenomena of the heavens and of our sea by the force of gravity, but I have not yet assigned a cause to gravity. Indeed, this force arises from some cause that penetrates as far as the centers of the sun and planets without any diminution of its power to act, and that acts not in proportion to the quantity of*

> *the* surfaces *of the particles on which it acts (as mechanical causes are wont to do) but in proportion to the quantity of* solid *matter, and whose action is extended everywhere to immense distances, always decreasing as the squares of the distances. Gravity toward the sun is compounded of the gravities toward the individual particles of the sun, and at increasing distances from the sun decreases exactly as the squares of the distances as far as the orbit of Saturn, as is manifest from the fact that the aphelia of the planets are at rest, and even as far as the farthest aphelia of the comets, provided that those aphelia are at rest. I have not as yet been able to deduce from phenomena the reason for these properties of gravity, and I do not "feign" hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy. In this experimental philosophy, propositions are deduced from the phenomena and are made general by induction. The impenetrability, mobility, and impetus of bodies, and the laws of motion and the law of gravity have been found by this method. And it is enough that gravity really exists and acts according to the laws that we have set forth and is sufficient to explain all the motions of the heavenly bodies and of our sea.*
>
> —Isaac Newton

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