First Order Extended Gravity and the Dark Side of the Universe: the General Theory

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Abstract. General Relativity is not the definitive theory of Gravitation due to several shortcomings which are coming out both from theoretical and experimental viewpoints. At large scales (astrophysical and cosmological scales) the attempts to match it with the today observational data lead to invoke Dark Energy and Dark Matter as the bulk components of the cosmic fluid. Since no final evidence, at fundamental level, exists for such ingredients, it is clear that General Relativity presents shortcomings at infrared scales. On the other hand, the attempts to formulate theories more general than the Einstein one give rise to mathematical difficulties that need workarounds which, in turn, generate problems from the interpretative viewpoint. We present here a completely new approach to the mathematical objects in terms of which a theory of Gravitation may be written in a first-order (*à la* Palatini) formalism, and introduce the concept of **Dark Metric** which could completely bypass the introduction of disturbing concepts as Dark Energy and Dark Matter.

INTRODUCTION

Einstein's General Relativity (**GR**) is a self-consistent theory which dynamically describes space, time and matter under the same standard. The result is a deep and beautiful scheme which, starting from some first principles, is capable of explaining a huge number of gravitational phenomena, ranging from laboratory up to cosmological scales. Its predictions are well tested at Solar System scales and give rise to a comprehensive cosmological model that agrees with Standard Model of particles, with recession of galaxies, cosmic nucleosynthesis and so on. Despite of these good results the recent advent of the so-called *Precision Cosmology* and several tests coming from the Solar System outskirts (e.g. the Pioneer Anomaly), the self-consistent scheme of GR seems to disagree with an increasingly high number of observational data, as *e.g.* those coming from IA-type Supernovae, used as standard candles, large scale structure ranging from galaxies up to galaxy superclusters and so on. Furthermore, being not renormalizable, it fails to be quantized in any "classical" way (see [1]). In other words, it seems then, from ultraviolet up to infrared scales, that GR is not and cannot be the definitive theory of Gravitation also if it successfully addresses a wide range of phenomena. Many attempts have been therefore made both to recover the validity of GR at all scales, on the one hand, and to produce theories that suitably generalize the Einstein one, on the other hand. In order to interpret a large number of recent observational data inside

the paradigm of GR, the introduction of Dark Matter (DM) and Dark Energy (DE) has seemed to be necessary: the price of preserving the *simplicity* of the Hilbert Lagrangian has been, however, the introduction of rather odd-behaving physical entities which, up to now, have not been revealed by any experiment at fundamental scales. In other words we are observing the large scale effects of missing matter (DM) and the accelerating behavior of the Hubble flow (DE) but no final evidence of these ingredients exists, if we want to deal with them under the standard of quantum particles or quantum fields. In Section 3 we shall argue whether, after all, it is really preferable the use of the *simplest* Lagrangian. An opposite approach resides in the so-called Non-Linear Theories of Gravity (NLTGs), that have been also investigated by many authors, in connection with Scalar-Tensor Theories (STTs). In this case, no ill-defined ingredients have to be required, at the price of big mathematical complications. None of the many efforts made up to now to solve this problem (see later) seem to be satisfactory from an interpretative viewpoint. What we shortly present here is a completely new approach to the mathematical objects in terms of which a theory of Gravitation may be written, whereby Gravity is encoded from the very beginning in a (symmetric) linear connection in SpaceTime. At the end we shall nevertheless conclude that, although the gravitational field is a linear connection, the fundamental field of Gravity turns out *a posteriori* to be still a metric, but not the "obvious" one given from the very beginning (which we shall therefore call *"apparent metric"*). Rather we shall show the relevance of another metric, ensuing from the gravitational dynamics, that we shall call **Dark Metric** since we claim it being a possible source of the *apparently* "Dark Side" of our Universe which reveals, at large scales, as missing matter (in clustered structures) and accelerating behavior (in the Hubble fluid). To complete our program, we need first to recall some facts regarding different (relativistic) theories of Gravitation. This will not be an historical *compendium*, but just a collection of speculative hints useful to our aims.

HISTORICAL REMARKS

Einstein devoted more than ten years (1905–1915/1916) to develop a theory of Gravitation based on the following requirements (see [3]): **principle of equivalence** (Gravity and Inertia are indistinguishable; there exist observers in free fall, *i.e.* inertial motion under gravitational pull); **principle of relativity** (SR holds pointwise; the structure of SpaceTime is pointwise Minkowskian); **principle of general covariance** ("democracy" in Physics); **principle of causality** (all physical phenomena propagate respecting the light-cones). Einstein, who was also deeply influenced by Riemann's teachings about the link between matter and curvature, decided then to describe Gravity by means of a (dynamic) SpaceTime *M* endowed with a dynamic Lorentzian metric *g*. This appeared to be a good choice for a number of reasons: a metric is the right tool to define measurements (rods & clocks); the geodesics of a metric are good mathematical objects to describe the free fall; a Lorentzian manifold is pointwise Minkowskian; it is suitable to be the domain of tensor fields; is compatible with a light-cones structure. And, after all, at that time, there was no other geometrical field that Einstein could use to define the curvature of a differentiable manifold!

Following this approach, Einstein deduced his famous equations:

$$G_{\mu\nu} :\equiv R_{\mu\nu} - \frac{1}{2} R(g) \, g_{\mu\nu} = 8\pi \, G \, T_{\mu\nu} \, .$$

A linear concomitant of the Riemann tensor of g, nowadays called the Einstein tensor $G_{\mu\nu}$, equals the stress-energy tensor $T_{\mu\nu} :\equiv \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ that reflects the properties of matter. Here $R_{\mu\nu}$ is the Ricci tensor of the metric g and $R(g) :\equiv g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature of the metric, while $L_{\text{mat}} \equiv \mathscr{L}_{\text{mat}} \, ds$ is the matter Lagrangian and G is the coupling constant. In other words, the distribution of matter influences Gravity through 10 second-order field equations. Their structure, in a sense and *mutatis mutandis*, is the same as Newton second law of Dynamics: no forces means geodesic motion, while the effects of sources are to produce curvature (just in motion in the Newtonian case, where the Space and Time are fixed and immutable; both in the structure of SpaceTime and in its motions in the Einstein case). GR has been a success: it admits an elegant and very *simple* Lagrangian formulation ($L_{\text{H}} :\equiv R(g) \sqrt{g} \, ds$) and most of its predictions have been soon experimentally verified and these have remained valid for many years after its introduction. So there was no reason for Einstein to be unhappy with his beautiful creation, at least for some time. In GR, is g the gravitational field? Einstein knew that it is not, since g is a tensor, while the principle of equivalence holds true and implies that there exist frames in which the gravitational field can be inertially switched off, while a tensor cannot be set to vanish in a frame, if it does not vanish in all frames. Free fall is, in fact, described by the geodesics of (M, g):

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}(g)\dot{x}^\mu \dot{x}^\nu = 0 \tag{1}$$

and Einstein himself argued that the right objects to represent the gravitational field have to be the Christoffel symbols $\Gamma^\lambda_{\mu\nu}(g)$; the metric g is just the potential of the gravitational field, but being Christoffel symbols algorithmically constructed out of g, the metric remains the fundamental variable: g gives rise to the gravitational field, to causality, to the principle of equivalence as well as to rods and clocks. In 1917, working on the theory of "parallelism" in manifolds, T. Levi-Civita understood that parallelism and curvature are non-metric features of space, but rather features of "affine" type, having to do with "congruences of privileged lines" (see [4]). Generalizing the case of Christoffel symbols $\Gamma^\lambda_{\mu\nu}(g)$ of a metric g, Levi-Civita introduced the notion of **linear connection** as the most general object $\Gamma^\lambda_{\mu\nu}$ such that the equation of geodesics (autoparallel curves, in fact)

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \tag{2}$$

is generally covariant. This revolutionary idea (that stands in fact at the heart of Non-Euclidean Geometries) was immediately captured by Einstein, who, unfortunately, did not further use it up to its real strength. We shall come back later on this topic, as this work is strongly based on it. Even if it was clear to Einstein that Gravity induces "freely falling observers" and that the principle of equivalence selects, in fact, an object that cannot be a tensor, since it is capable of being "switched off" and set to vanish at least in a point, he was obliged to choose it under the form of the linear connection $\Gamma_{\text{LC}}(g)$, given

locally by Christoffel symbols of g, now called the Levi-Civita connection (of g), fully determined by the metric structure itself. Einstein, for obvious reasons, was very satisfied of having reduced all SpaceTime structure and Gravity into a single geometrical object. Still, in 1918, H. Weyl tried (see [5]) to unify Gravity with Electromagnetism, using for the first time a linear connection defined over SpaceTime, assumed as a dynamical field non-trivially depending on a metric. Weyl's idea failed because of a wrong choice of the Lagrangian and few more issues, but it generated however a keypoint: connections may have a physically interesting dynamics. Einstein soon showed a great interest in Weyl's idea. He too began to play with connections, in the obsessed seek for the "geometrically" Unified Theory. But he never arrived to "dethronize" g in the description of the gravitational field. Probably, in some moments, he was not so happy with the fact that the gravitational field is not the fundamental object, but just a by-product of the metric; however, he never really changed his mind about the role of g. In 1925 Einstein constructed a theory that depends on a metric g and a symmetric linear connection Γ, to be varied independently (the so-called, because of a misunderstanding with W. Pauli, **Palatini method**; see [6]); he defined in fact a Lagrangian theory in which the gravitational Lagrangian is

$$L_{\text{PE}} :\equiv R(g,\Gamma)\,\sqrt{g}\,ds, \tag{3}$$

There are now $10 + 40$ independent variables and the field equations, in vacuum, are:

$$R_{(\mu\nu)} - \frac{1}{2}R(g,\Gamma)g_{\mu\nu} = 0$$
$$\nabla^{\Gamma}_{\alpha}(\sqrt{g}g^{\mu\nu}) = 0$$

$$\tag{4}$$

where $R_{(\mu\nu)}$ is the symmetric part of the Ricci tensor of Γ $R_{\mu\nu}(\Gamma,\partial\Gamma)$ and ∇^{Γ} denotes covariant derivative with respect to Γ.

Since the dimension of spacetime is greater than 2, the second field equation $(4)_2$ constrains the connection Γ, which is *a priori* arbitrary, to coincide *a posteriori* with the Levi-Civita connection of the metric g (Levi-Civita theorem). By substituting this information into the first field equation $(4)_1$, the vacuum Einstein equation for g is eventually obtained. In Palatini formalism, the metric g determines a priori rods & clocks, while the connection Γ the free fall, but since *a posteriori* the same result of GR is found, Einstein soon ceased to show a real interest in this formalism. The situation does not change if matter is present through a matter Lagrangian L_{mat} (independent of Γ but just depending on g and other external matter fields), that generates an energy-momentum tensor $T_{\mu\nu}$ as $T_{\mu\nu} :\equiv \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$. If the total Lagrangian is then assumed to be $L_{\text{tot}} :\equiv L_{\text{PE}} + L_{\text{mat}}$ field equation $(4)_1$ are reflected by

$$R_{\mu\nu} - \frac{1}{2}R(g,\Gamma)g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5}$$

and again $(4)_2$ implies, *a posteriori*, that (5) reduces eventually to Einstein equations in presence of matter.

Let us also emphasize that the dynamical coincidence between Γ and the Levi-Civita connection of g is entirely due to the particular Lagrangian considered by Einstein,

which is the *simplest*, but not the only possible one! Furthermore, it seems to us that Einstein did not fully recognize that the Palatini method privileges the affine structure with respect to the metric structure. Notice that, in this case (i.e. in Palatini formalism), the relations

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda}(g) \tag{6}$$

are field equations: the fact that Γ is the Levi-Civita connection of g is no longer an assumption *a priori* but it is the dynamical outcome of field equations!

THE GEOMETRIC STRUCTURE OF GRAVITATIONAL THEORIES

Let us begin this Section with a digression of fundamental importance in our schema. Thanks to the work of Levi-Civita and Einstein we know that every manifold having to do with gravity may be endowed with at least two a priori distinct structures:

- A *Riemannian metric structure*, defined by a (Lorentzian) metric tensor. It is responsible for the definition of (pseudo)distances and angles. It selects, on the space on which is defined, a class of hyper-surfaces as the level sets of the distance function (the analogous of circles in the ordinary Plane Geometry) and a class of curves, called the *geodesics of the metric*, or g-geodesics in this report, defined as the stationary (minimum) length paths connecting pairs of points.

- An *affine structure* defined by an affine connection. It is responsible for the parallel transport along curves through the definition of the notion of covariant derivation. It selects, on the space on which is defined, the notion of straightness of lines, i.e. a class of self-parallel lines (curves whose tangent vector is parallel transported by the connection, the analogous of straight lines in the ordinary Plane Geometry) called the *geodesics of the connection*, or Γ-*geodesics* in this work. As it is well known, on any such curve the notion of arc length is defined (independently of a metric).

  It is crucial to understand that, since the metric and the affine structures are a priori independent, the *g-geodesics* and the Γ-*geodesics* are, in principle very different. In exactly the same way, an arc length of a Γ-*geodesics* may not coincide with the distance of its extrema. This abstraction process is quite difficult to be performed, since the Geometry we are used to does not need it. Nevertheless it is a great opportunity, as we shall show in the sequel. As mathematical physicists we should have seen this kind of picture more than once....! In ordinary Euclidean Geometry, the metric and the affine structures (which together correspond to the well-known compass & unmarked straightedge Geometry) are, actually, deeply intertwined. A strong link between these two structures is in fact set by the *simplest* variational principle: straight lines are the shortest path between any two points. This is exactly to say: Γ-*geodesics* coincide with the *g-geodesics*. Formulated in the latter way, this situation is not peculiar of the ordinary Euclidean Geometry. On the contrary, it applies to all Euclidean and Non-Euclidean Geometries. This is why we have to force our *intuitus* to accept the more general case of geometries where Γ-*geodesics* do differ from the *g-geodesics*. With these considerations in

mind, let us reconsider the Palatini method we introduced in the previous Section. We believe it is clear enough that it can be considered an attempt of Einstein to force his *intuitus*. Not the first of his life. But this attempt, so to say, failed. We have already observed that the reason of this failure is entirely due to the *simplicity* of the Hilbert Lagrangian. This is why we regard to more general theories as an opportunity, not only as a complication. Before to delve into these more general theories, let us remark a final point about the metric and the affine structures. From a purely geometrical viewpoint, they stand on an equal footing. But from a physical point of view the situation is different. The fundamental principle of Newtonian Physics (the First Law, *i.e.* the Principle of Inertia) selects, in fact, the straight lines as the more fundamental structure: in absence of forces motions are rectilinear and uniform, *i.e.* 4-dimensionally straight. Circles instead limit their role to the definition of space distances. In GR something similar happens with the Principle of Equivalence (which in some sense is the generalization of the Newtonian Principle of Inertia): it selects the geodesics, the Γ-*geodesics*, as the most important lines from a gravitational point of view.

EXTENDED THEORIES OF GRAVITY

All that said, we believe we should first seriously reconsider NLTGs, without being unsensitive with respect to the appeal of *simplicity*, in the spirit of Occam Razor. This is why we begin to restrict ourselves to the first level of generalization of GR, the so-called $f(R)$-**theories** of metric type (see e.g. [7] for a review of the results concerning these theories). Here f denotes any "reasonable" function of one-real variable. The Lagrangian is assumed to be

$$L_{\mathrm{NL}}(g) :\equiv f(R(g))\sqrt{g}ds \tag{7}$$

where $R(g) = g^{\mu\nu}R_{\mu\nu}(g)$ is the scalar curvature of (M,g).
Of course, from $f(R)$-theories, we know that GR is retrieved in, and only in, the particular case $f(R) \equiv R$, i.e. if and only if the Lagrangian is linear[1] in R.
Let us recall here just a few keypoints on metric $f(R)$-theories.
When treated in the purely metric formalism, these theories are mathematically much more complicated than GR. These theories do in fact produce field equations that are of the fourth order in the metric:

$$f'(R(g))R_{\mu\nu}(g) - \frac{1}{2}f(R(g))g_{\mu\nu} - \underbrace{(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\,f'(R(g))}_{4^{\mathrm{th}}\text{ order term}} = 8\pi\,G\,T_{\mu\nu} \tag{8}$$

where f' denotes the derivative of f with respect to its real argument. This is something that cannot be accepted if one believes that physical laws should be governed

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[1] Of corse if $f(R) = R + \Lambda$ we have the Einstein Equations with a cosmological constant Λ.

by second order equations. In (8) we see a second order part that resembles Einstein tensor (and reduces identically to it if and only if $f(R) = R$, i.e. if and only if $f'(R) = 1$) and a fourth order "curvature term" (that again reduces to zero if and only if $f(R) = R$).

A first workaround that was suggested long ago to this problem is to push the 4th order part $(\nabla_\mu \nabla_v - g_{\mu v} \Box) f'(R(g))$ to the r.h.s. of these equations. This lets us to interpret it as an "extra gravitational stress" $T^{\text{curv}}_{\mu v}$ due to higher-order curvature effects, much in the spirit of Riemann. In any case, however, the fourth order character of these equations makes them unsuitable under several aspects, so that they were eventually abandoned for long time and only recently they have regained interest (see [7] and references therein).

A second way to tackle the problem has been proposed in 1987 (see [8], based on earlier work by the same authors [9], together with the references quoted therein). Notice that these are the first papers where the Legendre transformation that introduces an extra scalar field has been ever considered in literature (it has been later "re-discovered" by other authors), so that its priority should always be appropriately quoted when dealing with "metric" $f(R)$-theories. This is a method *à la* Hamilton, in which, whenever one has a non-linear gravitational metric Lagrangian of the most general type $L_{\text{GNL}}(g) :\equiv f(g, Ric(g))$, one defines a *second* metric p as[2]

$$p_{\mu v} :\equiv \frac{\partial L_{\text{grav}}}{\partial R_{\mu v}} \ . \tag{9}$$

In this way the second metric p, in fact a non-degenerate metric for f nowhere vanishing, a canonically conjugated momentum for g, is a function of g together with its first and second derivatives, since it is a function of g and $Ric(g)$, the Ricci tensor of g. Notice that this leads to two equations of the second order in g and p, as Hamilton method always halves the order of the equations by doubling the variables. Following this method in the simpler $f(R)$ case one gets that the "auxiliary" metric p is related to the original one g by a conformal transformation:

$$p \equiv \phi g, \qquad \phi :\equiv f'(R(g)). \tag{10}$$

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[2] This idea corresponds to an Einstein's attempt, dating back to 1925, to construct a "purely affine" theory (see [10]), i.e. a theory in which the only dynamical field is a linear connection. In this theory no metric is given from the beginning, but since it is obviously necessary to have a metric, the problem arise of how to construct it out of a connection. Einstein first tried to define the metric as the symmetric part of the Ricci tensor constructed out of the connection. But this idea could not work (unless for quadratic Lagrangians). A.Eddington then proposed the recipe (9). In this way Einstein and Eddington obtained a theory that reproduces GR, without introducing anything new. That is why Einstein eventually abandoned it too. On these purely affine theories see also [11], where J. Kijowski correctly pointed out that in the purely affine framework the prescription (9) of Einstein and Eddington is nothing but the assumption that the metric can be considered as a momentum canonically conjugated to the connection. Of corse p is a true metric if it is nondegenerate, something that is always true in an open set of the space of all solutions and for a ÒgenericÓ function f.

The Lagrangian equations (8) are then rewritable as a Hamiltonian system:

$$\text{Ein}(p) = T_{\text{mat}} + T_{\text{KGnl}} \tag{11}$$

$$\text{KGnl}(\phi) = 0 \tag{12}$$

where KGnl means non-linear Klein-Gordon (because of a potential depending on f; see [7], [8], [9] for details). Rewritten in this form, the theory has now two variables: the "auxiliary" metric p (or the original one g) and the scalar field ϕ. This is why these theories are called **Scalar-Tensor Theories**. For more details, and in particular for their application in Cosmology and Extragalactic Astrophysics, see, *e.g.*, [7] and the references quoted therein. Notice that [8], [9] and all subsequent literature left in fact open a few fundamental problems: *Who really are the second metric p and the scalar field ϕ? How to interpret them (the scalar field ϕ survives even in vacuum)? And...what about the original metric g?*
Fortunately there is a third method to solve the problem.

THE PALATINI APPROACH AND THE DARK METRIC

The third method anticipated at the end of the previous Section is the Palatini method applied to the case of $f(R)$-theories. Now SpaceTime is no longer a couple (M, g) but rather a triple (M, g, Γ), with Γ symmetric for simplicity and convenience. The Lagrangian is assumed to be the non-linear Palatini-Einstein Lagrangian

$$L_{\text{NLPE}}(g, \Gamma) :\equiv f(R(g, \Gamma))\sqrt{g}ds \tag{13}$$

with $R(g, \Gamma) :\equiv g^{\mu\nu}R_{\mu\nu}(\Gamma, \partial\Gamma)$ and f "reasonable." Field equations (4) are now replaced by the following:

$$f'(R(g, \Gamma))R_{(\mu\nu)} - \frac{1}{2}f(R(g, \Gamma))g_{\mu\nu} = G\,T_{\mu\nu} \tag{14}$$

$$\nabla_\alpha^\Gamma(f'(R(g, \Gamma))\sqrt{g}\,g^{\mu\nu}) = 0 \tag{15}$$

that take into account a possible Lagrangian of the type $L_{\text{mat}} = L_{\text{mat}}(g, \psi)$, with ψ arbitrary matter fields coupled to g alone (and *not* to Γ). Notice that $(15)_1$ reduces to (5) if and only if $f(R) \equiv R$. Notice also that the trace of equation $(15)_1$ gives

$$R(g, \Gamma)\,f'(R(g, \Gamma)) - \frac{m}{2}\,f(R(g, \Gamma)) = G\tau \tag{16}$$

being $\tau :\equiv g^{\mu\nu}\,T_{\mu\nu}$ the "trace" of the energy-momentum tensor. This equation has been called the **master equation** [13] and it is at the basis of a subtle discussion of "universality" of Einstein equations in non-linear special cases (i.e., when $\tau \equiv 0$). Notice also the analogy of (16) with the trace of (8), i.e.

$$f'(R(g))R(g) - \frac{m}{2}f(R(g)) + (m-1)\Box f'(R(g)) = G\tau \tag{17}$$

and notice that only in the peculiar case $f(R) = R$ they reduce to the *same* equation, namely (5). In all other cases (17) entails that non-linearity ($f' \neq 1$) produces, in the metric formalism, effects due to the scalar factor $f'(R)$, i.e. depending eventually on a scalar field tuned up by the curvature of Γ. Approaching $f(R)$-theories *à la* Palatini, we may now follow [13] step-by-step and make a number of considerations (well summarized also in the recent critical review [14]). At the end of these considerations we may conclude that:

1. When (and only when) $f(R(g,\Gamma)) = R(g,\Gamma)$ then GR is "fully" recovered for the given metric g.

2. For a generic $f(R(g,\Gamma))$, in presence of matter such that $g^{\mu\nu} T_{\mu\nu} = 0$ (and thence, in particular, in vacuum), the theory is still equivalent to GR for the given metric g with a "quantized" cosmological constant Λ and a modified coupling constant. In this case, in fact, the master equation (16) implies that the scalar curvature $R(g,\Gamma)$ has to be a suitable constant, possibly and usually not unique but always chosen in a set that depends on f, so that $(15)_2$ still implies (6) with an additional cosmological term $\Lambda g_{\mu\nu}$.

3. For a generic $f(R(g,\Gamma))$ but in presence of matter such that $g^{\mu\nu} T_{\mu\nu} \neq 0$ one can solve the master equation (16), for $m \neq 2$, and obtain $R(g,\Gamma)$ as a function $R(\tau)$ of the given trace τ. Then, knowing f, one gets implicitly $f(R(g,\Gamma)) = f(\tau)$ and $f'(R(g,\Gamma)) = f'(\tau)$, so that equation $(15)_2$ tells us that Γ is forced to be the Levi-Civita connection $\Gamma_{\mathrm{LC}}(h)$ of a new metric h, conformally related to the original one g by the relation

$$ h_{\mu\nu} :\equiv f'(\tau)\, g_{\mu\nu} = f'(R(g,\Gamma)) g_{\mu\nu} \,. \tag{18} $$

Then, using again equation $(15)_1$, we see that the theory could be still rewritable as in (10) in a purely metric setting, but with far less interpretative problems, as we can immediately show. From a viewpoint "*à la Palatini*" in a genuine sense the method has in fact generated a completely new perspective. The remaining field equations $(15)_1$, in fact, are still equivalent to Einstein equations with matter (and cosmological constant) provided one changes the metric from g to h!

The most enlightening case is that of $f(R)$ with generic matter and $\tau \neq 0$. Here, in fact, the universality property (see again [13]) does not hold in his strict form, but in an interesting wider interpretation: the dynamics of the connection Γ still forces Γ itself to be the Levi-Civita connection of a metric, but not of the "original" metric g, which we shall prefer to call the **apparent metric** for a reason we clarify in a moment. Instead, the dynamics of Γ identifies a new metric h, conformally related to the apparent one g, which we call the **Dark Metric**. The Dark Metric h, we claim, could well be the true origin of the "Dark Side of the Universe"! The apparent metric g is in fact the one by means of which we perform measurements in our local laboratories. In other words, the metric g is the one we have to use every day to construct and read instruments (rods & clocks). This is why we like to call it the "apparent" metric. But we claim that the right metric we have to use as the fundamental object to describe Gravity is, by obvious reasons, the Dark Metric, since it is the one responsible for gravitational free fall through the identification

$\Gamma = \Gamma_{LC}(h)$. Notice, incidentally, that photon world-lines and causality are not changed, since the light-cones structure of g and h are the same by conformal invariance. In other words, in our laboratories we have to use the apparent metric g, but in our Gravity theories the dark one h. The translation from one "language" to the other is nothing but the conformal factor $f'(R) = f'(\tau)$, which manifestly depends on the theory and on its content in ordinary matter. Let us also notice explicitly that this in particular implies that if a certain metric h is expected to be a solution of a problem, from a theoretical point of view, it is rather important to look for h in experiments. Testing our theories with g, in a sense, is wrong, since it is the conformally related metric h to be searched for instead when dealing with Gravitation!

CONCLUDING REMARKS

The (unknown) conformal factor $\phi \equiv f'(R(g, \Gamma)) \equiv f'(\tau)$ has to be phenomenologically tested against observational data in order to find which is the (class of) Lagrangian(s) f that, given τ (*i.e.*, given the "visible matter"), allow one to interpret the "supposed Dark Matter" (and Energy) as a curvature effect [15]. In the other contribution by the same authors in these Proceedings, we will give hints on how DE (accelerated cosmic behavior) and DM (clustered structures) could be interpreted as Dark Metric effects according to the lines dveloped here.

REFERENCES

1. R. Utiyama, B. S. DeWitt, J. Math. Phys. **3** (1962), 608.
2. A. Einstein, *Zur Elektrodynamic der bewegter Körper*, Ann. der Physik **XVII** (1905), 891–921
3. J. Ehlers, F. A. E. Pirani, A. Schild, *The Geometry of Free Fall and the Light Propagation*, in "Studies in Relativity: Papers in honour of J. L. Synge" (1972), 63–84
4. T. Levi-Civita, *The Absolute Differential Calculus*, Blackie and Son (1929)
5. M. Ferraris, M. Francaviglia, C. Reina, *Variational Formulation of General Relativity from 1915 to 1925 "Palatini's Method" Discovered by Einstein in 1925* - Gen. Rel. Grav. **14** (1982) 243-254
6. Hermann Weyl, Ann. Phys. **59** (1919), 101
7. S. Capozziello, M. Francaviglia, *Extended Theories of Gravitation and Their Cosmological and Astrophysical Applications*, Gen. Rel. Grav. **40**, 2-3, (2008), 357–420
8. G. Magnano, M. Ferraris, M. Francaviglia, *Legendre Transformation and Dynamical Structure of Higher-Derivative Gravity*, Class. Quantum Grav. **7** (1990), 557-570
9. G. Magnano, M. Ferraris, M. Francaviglia, *Nonlinear Gravitational Lagrangians*, Gen. Rel. Grav. **19**(5) (1987), 465–479
10. A. Eddington, *The Mathematical Theory of Relativity*, Cambridge University Press (1923)
11. J. Kijowski, *On a purely affine formulation of general relativity*, Part II Proceedings of the Conference Held at Salamanca September 10-Ð14 (1979); Edited by P. L. García, A. Pérez-Rendón
12. M. Ferraris, M. Francaviglia, I. Volovich, *Universality of Einstein Equations*, Class. Quantum Grav. **11 6** (1994), 1505–1517
13. T. P. Sotiriou, V. Faraoni, *f(R) Theories of Gravity*, (2008), arXiv:0805.1726
14. A. Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, Oxford University Press (1983)
15. S. Capozziello, V.F. Cardone, A. Troisi, *Dark Energy and Dark Matter as curvature effects?* JCAP **08** (2006), 001–14