1996

Ricci Tensor of Diagonal Metric

K. Z. Win
University of Massachusetts - Amherst


This Article is brought to you for free and open access by the Astronomy at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Astronomy Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.
Ricci Tensor of Diagonal Metric

K. Z. Win
Department of Physics and Astronomy
University of Massachusetts
Amherst, MA 01003

Abstract
Efficient formulae of Ricci tensor for an arbitrary diagonal metric are presented.
Introduction

Calculation of the Ricci tensor is often a cumbersome task. In this note useful formulae of the Ricci tensor are presented in equations (1) and (2) for the case of the diagonal metric tensor. Application of the formulae in computing the Ricci tensor of $n$-sphere is also presented.

Derivation

The sign conventions and notation of Wald [1] will be followed. Latin indices are part of abstract index notation and Greek indices denote basis components. The Riemann curvature tensor $R_{\alpha \beta \gamma \delta}$ is defined by $\nabla_{\alpha} \nabla_{\beta} g_{\gamma \delta} - \nabla_{\alpha} \nabla_{\gamma} g_{\beta \delta} = R_{\alpha \beta \gamma \delta}$. The Ricci tensor in a coordinate basis is $R_{\lambda \nu} = \partial_{\rho} \Gamma^{\rho}_{\lambda \nu} - \partial_{\lambda} \Gamma^{\rho}_{\rho \nu} + \Gamma^{\sigma}_{\lambda \rho} \Gamma^{\rho}_{\sigma \nu}$ where the Christoffel symbols are $\Gamma^{\rho}_{\lambda \nu} = g^{\rho \sigma} (\partial_{\lambda} g_{\nu \sigma} + \partial_{\nu} g_{\lambda \sigma} - \partial_{\sigma} g_{\lambda \nu}) / 2$. For the rest of this note we assume that $g_{\rho \nu}$ is diagonal unless otherwise indicated. Also to avoid having to write down "$\mu \neq \nu$" repeatedly we will always take $\mu \neq \nu$ and no sum will be assumed on repeated $\mu$ and $\nu$.

We first note that $\Gamma^{\rho}_{\sigma \mu} = 0$ if $\mu \neq \sigma \neq \nu$. It means that at least two indices of $\Gamma$ must be the same for it to be nonvanishing. Also $\ln |g| = \sum_{\sigma=1}^{n} \ln |g_{\sigma \sigma}|$ where $g$ and $n$ are determinant of the metric and dimension of manifold, respectively. Then it is easy to show that $2 \Gamma^{\mu}_{\sigma \mu} = \partial_{\sigma} \ln |g_{\mu \mu}|$ for all $\sigma$ and that $2 \Gamma^{\nu}_{\mu \nu} = -g^{\mu \nu} \partial_{\nu} g_{\mu \mu}$. (Again $\mu \neq \nu$ and no sum on $\mu$ and $\nu$.) These two equations give

$$4 \Gamma^{\nu}_{\mu \nu} \Gamma^{\mu}_{\nu \nu} = g^{\nu \nu} (\partial_{\nu} g_{\mu \mu}) g^{\mu \mu} (\partial_{\mu} g_{\nu \nu}) = (\partial_{\mu} \ln |g_{\nu \nu}|) (\partial_{\mu} \ln |g_{\nu \nu}|) = 4 \Gamma^{\mu}_{\mu \nu} \Gamma^{\nu}_{\mu \nu}$$

Using above facts, we get, with each sum having upper limit $n$ and lower limit 1,

$$R_{\mu \nu} = \partial_{\nu} \Gamma^{\mu}_{\mu \nu} + \partial_{\nu} \Gamma^{\nu}_{\mu \nu} - (1/2) \partial_{\mu} \partial_{\nu} \ln |g| - \Gamma^{\mu}_{\mu \rho} \Gamma^{\rho}_{\mu \nu} - \Gamma^{\nu}_{\mu \rho} \Gamma^{\rho}_{\nu \mu} - \sum_{\sigma \neq \mu, \nu} \Gamma^{\sigma}_{\mu \rho} \Gamma^{\rho}_{\sigma \nu}$$

$$+ (1/2) \Gamma^{\mu}_{\mu \nu} \partial_{\nu} \ln |g| + (1/2) \Gamma^{\nu}_{\mu \nu} \partial_{\nu} \ln |g|$$

$$= (1/2) \partial_{\mu} \partial_{\nu} \ln |g|_{\mu \mu} g_{\nu \nu} - \Gamma^{\mu}_{\mu \mu} \Gamma^{\mu}_{\mu \nu} - \Gamma^{\nu}_{\mu \mu} \Gamma^{\mu}_{\mu \nu} - \Gamma^{\nu}_{\mu \mu} \Gamma^{\nu}_{\mu \nu} - \Gamma^{\nu}_{\mu \mu} \Gamma^{\nu}_{\mu \nu} - \sum_{\sigma \neq \mu, \nu} \Gamma^{\sigma}_{\mu \rho} \Gamma^{\rho}_{\sigma \nu}$$

$$+ \Gamma^{\mu}_{\mu \nu} \partial_{\mu} \ln \sqrt{|g_{\mu \mu}|} + \Gamma^{\nu}_{\mu \nu} \partial_{\nu} \ln \sqrt{|g_{\nu \nu}|} + \Gamma^{\nu}_{\mu \nu} \partial_{\nu} \ln \sqrt{|g_{\mu \mu}|} + \Gamma^{\nu}_{\mu \nu} \partial_{\nu} \ln \sqrt{|g_{\nu \nu}|}$$

$$+ (1/2) \Gamma^{\mu}_{\mu \nu} \partial_{\nu} \ln \frac{g}{g_{\mu \mu} g_{\nu \nu}} + (1/2) \Gamma^{\nu}_{\mu \nu} \partial_{\nu} \ln \frac{g}{g_{\mu \mu} g_{\nu \nu}}$$
\begin{align*}
&(1/2)\partial_\mu \partial_\nu \ln \left| \frac{g_{\mu\nu}g_{\nu\nu}}{g} \right| - \Gamma^\mu_{\mu\nu} \Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\mu\nu} \Gamma^\mu_{\mu\nu} - \Gamma^\nu_{\mu\nu} \Gamma^\mu_{\nu\nu} - \sum_{\sigma \neq \mu, \nu} \Gamma^\sigma_{\sigma\mu} \Gamma^\sigma_{\sigma\nu} \\
&\quad + \Gamma^\mu_{\mu\nu} + \sum_{\sigma \neq \mu, \nu} \Gamma^\sigma_{\sigma\mu} \Gamma^\sigma_{\sigma\nu} \\
&\quad + \Gamma^\mu_{\mu\nu} + \Gamma^\nu_{\mu\nu} \Gamma^\mu_{\mu\nu} + \Gamma^\nu_{\nu\nu} \Gamma^\mu_{\nu\nu} \\
&\quad + (1/2)\Gamma^\mu_{\mu\nu} \partial_\mu \ln \left| \frac{g}{g_{\mu\mu}g_{\nu\nu}} \right| + (1/2)\Gamma^\nu_{\mu\nu} \partial_\nu \ln \left| \frac{g}{g_{\mu\mu}g_{\nu\nu}} \right| \\
&\quad + (1/2)\sum_{\sigma \neq \mu, \nu} \Gamma^\sigma_{\sigma\mu} \partial_\mu \ln \left| \frac{g}{g_{\mu\mu}g_{\nu\nu}} \right| + (1/2)\sum_{\sigma \neq \mu, \nu} \Gamma^\sigma_{\sigma\nu} \partial_\nu \ln \left| \frac{g}{g_{\mu\mu}g_{\nu\nu}} \right|
\end{align*}

or (remember \(\mu \neq \nu\))

\[
4R_{\mu\nu} = (\partial_\mu \ln |g_{\nu\nu}| - \partial_\nu \ln |g_{\mu\mu}|) + (\mu \leftrightarrow \nu) - \sum_{\sigma \neq \mu, \nu} \partial_\mu \ln |g_{\sigma\sigma}| \partial_\nu \ln |g_{\sigma\sigma}| \tag{1}
\]

where \((\mu \leftrightarrow \nu)\) stands for preceding terms with \(\mu\) and \(\nu\) interchanged.* A number of useful facts follow from the above formula. First of all \(R_{\mu\nu} = 0\) if \(x_\mu\) is an ignorable coordinate because \(\partial_\mu\) is in each term of the formula. If there are \(m\) ignorable coordinates then maximum number of nonzero off-diagonal components of the Ricci tensor is \((n-m)(n-m-1)/2\). Thus a sufficient condition for the Ricci tensor to be diagonal is that the (diagonal) metric be a function only of one coordinate. The Robertson-Walker metric with flat spatial sections, \(ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\), satisfies this condition and its Ricci tensor is consequently diagonal. If the (diagonal) metric is a function only of two coordinates, say \(x_\mu\) and \(x_\nu\), then all off diagonal components of the Ricci tensor except \(R_{\mu\nu}\) are zero. For example, for Schwartzschild metric \(ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\) only \(R_{\theta\theta}\) needs to be explicitly calculated, all other being zero. It is easily checked using equation (1) that \(R_{\theta\theta}\) is also zero. Symmetry arguments can also be given† as to why the Ricci tensor is diagonal in this case. Secondly second derivative terms are identically zero whenever each component of the metric is a product of functions of one single coordinate i.e. if, for all \(\mu\), \(g_{\mu\mu} = \prod_{l=1}^n f^l_\mu(x_l)\) where some of \(f^l_\mu\) may be one. A final trivial corollary

* Alternative forms of equation (1) are

\[
4R_{\mu\nu} = \sum_{\sigma \neq \mu, \nu} \left[(\partial_\mu \ln |g_{\nu\nu}| - \partial_\nu \ln |g_{\mu\mu}|) + (\mu \leftrightarrow \nu) - \partial_\mu \ln |g_{\sigma\sigma}| \partial_\nu \ln |g_{\sigma\sigma}|\right] \\
8R_{\mu\nu} = \sum_{\sigma \neq \mu, \nu} \left[(\partial_\mu \ln |g_{\nu\nu}^2|_{\sigma\sigma} - 2\partial_\mu) \partial_\nu \ln |g_{\sigma\sigma}| + (\mu \leftrightarrow \nu)\right]
\]

† Page 178 of Weinberg [2].
of equation (1) is that the Ricci tensor is diagonal in 2-dimensions. This fact also follows
trivially from the fact that in 2-dimensions, the Ricci tensor is the metric tensor (not
necessarily diagonal) up to a factor of a scalar function. When \( n = 3 \), it is easy to find an
example of a diagonal metric which results in a non-diagonal Ricci tensor. For example,
\( ds^2 = dx^2 + xdy^2 + ydz^2 \) gives \( 4R_{xy} = 1/xy \).

We now calculate the diagonal components of Ricci tensor. We have

\[
R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial^2_{\mu} \ln \sqrt{|g|} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\mu\nu} \partial_\rho \ln \sqrt{|g|}
\]

\[
= \partial^2_\mu \ln \sqrt{|g_{\mu\nu}|} + \sum_{\sigma \neq \mu} \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial^2_{\mu} \ln \sqrt{|g|} - \Gamma^\mu_{\mu\rho} \Gamma^\rho_{\sigma\nu}
\]

\[
- \sum_{\sigma \neq \mu} \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu} + \Gamma^\mu_{\mu\rho} \partial_\rho \ln \sqrt{|g|} + \sum_{\sigma \neq \mu} \Gamma^\sigma_{\mu\rho} \partial_\sigma \ln \sqrt{|g|}
\]

\[
= (1/2) \partial^2_{\mu} \ln \frac{|g_{\mu\nu}|}{g} + \sum_{\sigma \neq \mu} \partial_\sigma \Gamma^\sigma_{\mu\nu} - (\Gamma^\mu_{\mu\rho})^2 - \sum_{\sigma \neq \mu} \Gamma^\mu_{\mu\rho} \Gamma^\rho_{\sigma\nu} - \sum_{\sigma \neq \mu} \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu}
\]

\[
- \sum_{\sigma \neq \mu} \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu} + \Gamma^\mu_{\mu\rho} \partial_\rho \ln \sqrt{|g|} + \sum_{\sigma \neq \mu} \Gamma^\sigma_{\mu\rho} \partial_\sigma \ln \sqrt{|g|}
\]

\[
= (1/4) (\partial_\mu \ln |g_{\mu\nu}| - 2\partial_\mu) \partial_\mu \ln \frac{g}{g_{\mu\nu}} + \sum_{\sigma \neq \mu} \left[ \partial_\sigma \Gamma^\sigma_{\mu\nu} - 2\Gamma^\mu_{\mu\rho} \Gamma^\rho_{\sigma\nu} - (\Gamma^\sigma_{\mu\nu})^2 \right]
\]

\[
+ \Gamma^\sigma_{\mu\rho} \partial_\sigma \ln \sqrt{|g|}
\]

\[
= (1/4) (\partial_\mu \ln |g_{\mu\nu}| - 2\partial_\mu) \partial_\mu \ln \frac{g}{g_{\mu\nu}} + \sum_{\sigma \neq \mu} \left[ - (1/2) \partial_\sigma (g^{\sigma\rho} \partial_\rho g_{\mu\nu}) \right.
\]

\[
+ g^{\sigma\rho} (\partial_\sigma g_{\mu\nu}) \partial_\sigma \ln \sqrt{|g_{\mu\nu}|} - \left( \partial_\mu \ln \sqrt{|g_{\sigma\nu}|} \right)^2 - (g^{\sigma\rho}/2)(\partial_\sigma g_{\mu\nu}) \partial_\sigma \ln \sqrt{|g|}
\]

\[
= \frac{1}{4} (\partial_\mu \ln |g_{\mu\nu}| - 2\partial_\mu) \partial_\mu \ln \frac{g}{g_{\mu\nu}} - \frac{1}{4} \sum_{\sigma \neq \mu} \left[ 2\partial_\sigma (g^{\sigma\rho} \partial_\rho g_{\mu\nu}) + (\partial_\mu \ln |g_{\sigma\nu}|)^2 
\]

\[
+ g^{\sigma\rho} (\partial_\sigma g_{\mu\nu}) \partial_\sigma \ln \frac{|g|}{g_{\mu\nu}^2} \right]
\]

or

\[
4R_{\mu\nu} = (\partial_\mu \ln |g_{\mu\nu}| - 2\partial_\mu) \partial_\mu \ln \frac{g}{g_{\mu\nu}} - \sum_{\sigma \neq \mu} \left[ (\partial_\mu \ln |g_{\sigma\nu}|)^2 + (\partial_\sigma \ln \frac{|g|}{g_{\mu\nu}^2} + 2\partial_\sigma) g^{\sigma\rho} \partial_\rho g_{\mu\nu} \right]
\]

where \( 2\partial_\sigma \) acts everything on its right. The efficiency of this formula is obvious for
Schwarzschild metric given earlier. To compute \( R_{tt} \) we note that all terms with \( \partial_t \) are zero.
Thus $4R_{tt} = - \sum_{\sigma \neq t} (\partial_\sigma \ln \frac{|g|}{g_{tt}^\sigma} + 2\partial_\sigma) (g^{\sigma\sigma} \partial_\sigma g_{tt})$. Because $g_{tt} = -B(r)$ only the $\sigma = r$ term survives and we get, after very little computation, $R_{tt} = \frac{B'}{A} - \frac{B'}{4A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{B''}{2A}$ where prime indicates differentiation with respect to $r$.\footnote{Cf. page 178 of Weinberg and note that his $R_{abcd}$ is minus of Wald’s. Also see problem 6.2 of [1].} Other diagonal components are just as easy to compute.

**Application to $n$-sphere**

To illustrate the use of above formulae we compute Ricci tensor of $n$-sphere which in spherical coordinates has diagonal metric $g_{11} = 1$ and $g_{\mu+1,\mu+1} = g_{\mu\mu} \sin^2 x_\mu$ for $1 \leq \mu \leq n - 1$. First note that $\ln |g_{\mu\mu}| = \sum_{\nu=1}^{\mu-1} 2 \ln |\sin x_\nu|$ and thus $\partial_\rho g_{\mu\mu} = 0$ if $n \geq \rho \geq \mu \geq 1$. These facts will be used repeatedly in what follows. We first compute the off diagonal components of Ricci tensor. According to the general argument given earlier second derivative terms will be zero. Let $\mu > \nu$. Then equation (1) becomes

$$4R_{\mu\nu} = \sum_{\sigma > \mu} \left[ (\partial_\nu \ln |g_{\mu\mu}|) \partial_\mu \ln |g_{\sigma\sigma}| - (\partial_\mu \ln |g_{\sigma\sigma}|) \partial_\nu \ln |g_{\sigma\sigma}| \right]$$

$$= 4 \sum_{\sigma > \mu} (\cot x_\nu \cot x_\mu - \cot x_\mu \cot x_\nu) = 0$$

We next compute the diagonal components. We have, from equation (2),

$$4R_{\mu\mu} = - \sum_{\sigma > \mu} \left[ 2\partial_\mu^2 \ln |g_{\sigma\sigma}| + (\partial_\mu \ln |g_{\sigma\sigma}|)^2 \right] - \sum_{\sigma < \mu} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu\mu}^\sigma} + 2\partial_\sigma \right) g^{\sigma\sigma} \partial_\sigma g_{\mu\mu}$$

$$= \sum_{\sigma > \mu} \left( -2 \times 2\partial_\mu \cot x_\mu - 4 \cot^2 x_\mu \right) - \left( \partial_{\mu-1} \ln \frac{|g|}{g_{\mu\mu}^{\mu-1}} + 2\partial_{\mu-1} \right) g^{\mu-1,\mu-1} \partial_{\mu-1} g_{\mu\mu}$$

$$- \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}^{\sigma}} - 4\partial_\sigma \ln |\sin x_{\mu-1}| + 2\partial_\sigma \right) g^{\sigma\sigma} \partial_\sigma \left( g_{\mu-1,\mu-1} \sin^2 x_{\mu-1} \right)$$

$$= \sum_{\sigma > \mu} \left( \frac{4}{\sin^2 x_\mu} - \frac{4\cos^2 x_\mu}{\sin^2 x_\mu} \right)$$

$$- \left[ \sum_{\sigma > \mu-1} \partial_{\mu-1} \ln |g_{\sigma\sigma}| - 2\partial_{\mu-1} \ln |g_{\mu\mu}| + 2\partial_{\mu-1} \right] g^{\mu-1,\mu-1} \partial_{\mu-1} g_{\mu\mu}$$

$$- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}^{\sigma}} + 2\partial_\sigma \right) g^{\sigma\sigma} \partial_\sigma g_{\mu-1,\mu-1-1}$$
\[= 4(n - \mu) - \left[ (2 \cot x_{\mu-1} \sum_{\sigma > \mu-1} \right) - 4 \cot x_{\mu-1} + 2 \partial_{\mu-1} \right] g^{\mu-1,\mu-1} g_{\mu-1,\mu-1} \sin 2x_{\mu-1} \]

\[- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \]

\[= 4(n - \mu) - \left[ 2 \cot x_{\mu-1}(n + 1 - \mu) - 4 \cot x_{\mu-1} \right] \sin 2x_{\mu-1} - 2 \times 2 \cos 2x_{\mu-1} \]

\[- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \]

\[= 4(n - \mu) - 4 \left[ \cos^2 x_{\mu-1}(n - \mu + 1) - 2 \cos^2 x_{\mu-1} + \cos^2 x_{\mu-1} - \sin^2 x_{\mu-1} \right] \]

\[- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \]

\[= 4 \left[ n - \mu - \cos^2 x_{\mu-1}(n - \mu + 1) + 1 \right] \]

\[- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \]

\[= \sin^2 x_{\mu-1} \left[ 4(n - \mu + 1) - \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \right] \]

Thus

\[4R_{\mu+1,\mu+1} = \sin^2 x_{\mu} \left[ 4(n - \mu) - \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu,\mu} \right] \]

We now rewrite the expression inside the square brackets. Break up the sum into \( \sigma = \mu - 1 \) and \( \sigma < \mu - 1 \) and use \( g_{\mu,\mu} = g_{\mu-1,\mu-1} \sin^2 x_{\mu-1} \). Then

\[\left[ \right] = 4(n - \mu) - \left( \partial_{\mu-1} \ln \frac{|g|}{g_{\mu-1,\mu}} + 2 \partial_{\mu-1} \right) g^{\mu-1,\mu-1} \partial_{\mu-1} g_{\mu,\mu} \]

\[- \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} - 4 \partial_\sigma \ln |\sin x_{\mu-1}| + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu,\mu} \]

\[= 4(n - \mu) - \left( \partial_{\mu-1} \ln |g| - 2 \partial_{\mu-1} \ln |g_{\mu,\mu}| + 2 \partial_{\mu-1} \right) g^{\mu-1,\mu-1} g_{\mu-1,\mu-1} \sin 2x_{\mu-1} \]

\[- \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma \left( g_{\mu-1,\mu-1} \sin^2 x_{\mu-1} \right) \]

\[= 4(n - \mu) - \left[ \left( \sum_{\sigma > \mu-1} \partial_{\sigma} \ln |g_{\sigma\sigma}| \right) - 2 \times 2 \cot x_{\mu-1} + 2 \partial_{\mu-1} \right] \sin 2x_{\mu-1} \]

\[- \sin^2 x_{\mu-1} \sum_{\sigma < \mu-1} \left( \partial_\sigma \ln \frac{|g|}{g_{\mu-1,\mu-1}} + 2 \partial_\sigma \right) g^{\sigma \sigma} \partial_\sigma g_{\mu-1,\mu-1} \]
\[ =4(n - \mu) - [2 \cot x_{\mu - 1}(n - \mu + 1) - 4 \cot x_{\mu - 1}] \sin 2x_{\mu - 1} - 4 \cos 2x_{\mu - 1} \]

\[ - \sin^2 x_{\mu - 1} \sum_{\sigma < \mu - 1} \left( \partial_{\sigma} \ln \frac{|g|}{g_{\mu - 1, \mu - 1}^2} + 2 \partial_{\sigma} \right) g^{\sigma \sigma} \partial_{\sigma} g_{\mu - 1, \mu - 1} \]

\[ =4(n - \mu) - 4 \left[ \cos^2 x_{\mu - 1}(n - \mu + 1) - 2 \cos^2 x_{\mu - 1} + \cos^2 x_{\mu - 1} - \sin^2 x_{\mu - 1} \right] \]

\[ - \sin^2 x_{\mu - 1} \sum_{\sigma < \mu - 1} \left( \partial_{\sigma} \ln \frac{|g|}{g_{\mu - 1, \mu - 1}^2} + 2 \partial_{\sigma} \right) g^{\sigma \sigma} \partial_{\sigma} g_{\mu - 1, \mu - 1} \]

\[ =4\left[ n - \mu - \cos^2 x_{\mu - 1}(n + 1 - \mu) + \sin^2 x_{\mu - 1} + \cos^2 x_{\mu - 1} \right] \]

\[ - \sin^2 x_{\mu - 1} \sum_{\sigma < \mu - 1} \left( \partial_{\sigma} \ln \frac{|g|}{g_{\mu - 1, \mu - 1}^2} + 2 \partial_{\sigma} \right) g^{\sigma \sigma} \partial_{\sigma} g_{\mu - 1, \mu - 1} \]

\[ =4(n + 1 - \mu) \sin^2 x_{\mu - 1} - \sin^2 x_{\mu - 1} \sum_{\sigma < \mu - 1} \left( \partial_{\sigma} \ln \frac{|g|}{g_{\mu - 1, \mu - 1}^2} + 2 \partial_{\sigma} \right) g^{\sigma \sigma} \partial_{\sigma} g_{\mu - 1, \mu - 1} \]

\[ =4R_{\mu \mu} \]

Next

\[ 4R_{11} = -\sum_{\sigma > 1} \left[ 2 \partial_1^2 \ln |g_{\sigma \sigma}| + (\partial_1 \ln |g_{\sigma \sigma}|)^2 \right] = 4 \left( \frac{1}{\sin^2 x_1} - \cot^2 x_1 \right) \sum_{\sigma > 1} = 4(n - 1) \]

All preceding calculations amount to proving that \( R_{\rho \mu} = (n - 1)g_{\rho \mu} \). Let \( \{ v_\rho^a \} \) be the dual basis of the tangent space where \( \rho \) labels a particular basis vector. Then

\[ R_{ab} = R_{\rho \sigma} v_\rho^a v_\sigma^b = (n - 1)g_{\rho \sigma} v_\rho^a v_\sigma^b = (n - 1)g_{ab} \]

This fact about \( n \)-sphere is also provable using the machinery of symmetric spaces. See chapter 13 of [2].

Equations (1) and (2) were derived when attempt was made to establish any connection between diagonality of \( R_{\nu \sigma} \) and that of \( g_{\sigma \nu} \). Finally we remark that Dingle [3] has listed values of \( T^\nu_\nu \) for the case of 4-dimensional diagonal metric.

**Acknowledgements**

I would like to thank Professor Jennie Traschen for much encouragement and discussion and for useful comments on the manuscript.
References

