Binomial Expansion for $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2}$

The Maclaurin Series generated by

$$f(x) = (1+x)^n, n \leq 0,$$

is written as

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

$$\text{where} \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ and } 0! = 1, \binom{n}{0} = 1,$$

is an infinite series which converges only for $|x| < 1 \Leftrightarrow -1 < x < 1$.

This series is a binomial series where the expansion goes as follows:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \binom{n}{4} x^4 + \ldots$$

$$= \frac{n!}{0!(n-0)!} x^0 + \frac{n!}{1!(n-1)!} x^1 + \frac{n!}{2!(n-2)!} x^2 + \frac{n!}{3!(n-3)!} x^3 + \frac{n!}{4!(n-4)!} x^4 + \ldots$$

$$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \frac{n(n-1)(n-2)(n-3)}{4!} x^4 + \ldots$$

$$(1+x)^n = \quad = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \frac{n(n-1)(n-2)(n-3)}{24} x^4 + \ldots$$

$\Longrightarrow$

Let $n = -\frac{1}{2}$, $x = -\beta^2$ where $\mid \beta \mid = \mid \frac{v}{c} \mid < 1$, then

$$(1-\beta^2)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-\beta^2) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}(-\beta^2)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{6}(-\beta^2)^3 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{24}(-\beta^2)^4 + \ldots$$

$$(1-\beta^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \tfrac{5}{16}\beta^6 + \tfrac{35}{128}\beta^8 + \ldots$$

$$\Longrightarrow \quad \therefore \boxed{(1-\beta^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \frac{35}{128}\frac{v^8}{c^8} + \ldots}$$

Please note that this binomial expansion is integral for eventually deriving Einstein's $E = mc^2$ and hence is of vital importance!