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Variational Formulation of General Relativity from 1915 to 1925 "Palatini's Method" Discovered by Einstein in 1925

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Abstract

Among the three basic variational approaches to general relativity, the metric–affine variational principle, according to which the metric and the affine connection are varied independently, is commonly known as the "Palatini method." In this paper we revisit the history of the "golden age" of general relativity, through a discussion of the papers involving a variational formulation of the field problem. In particular we find that the original Palatini paper of 1919 was rather far from what is usually meant by "Palatini's method," which was instead formulated, to our knowledge, by Einstein in 1925.

§(1): *Introduction*

According to standard methods in mathematical physics, one of the more common procedures to derive Einstein's field equations for general relativity is to make use of a suitable variational principle.

In recent years, the variational approaches to general relativity have received renewed attention, especially owing to the attempts to construct consistent gauge theories of gravitation. Motivated by this, we began investigating the subject with the aim of collecting, reviewing, and clarifying the role of variational principles in the theories of gravitation. At the very beginning we have been faced with a historical problem concerning what is commonly known as "Palatini's variational method." Namely, we found that in the current literature there

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are many papers in which the original results by Palatini are incorrectly quoted. This is most probably due to the fact that Palatini's paper is written in Italian and may be not easily accessible to the whole scientific community.[1]

In looking for the sources of the confusion mentioned above, we found it interesting and stimulating to revisit the early developments of variational principles in the "golden age" of general relativity (i.e., in the years from 1915 to 1925). Incidentally, this review will answer the question about who first formulated the so-called Palatini method. We understand that the priority problem has no relevance to the scientific progress in itself. However, it involves, as we shall see later, substantial problems which, in our opinion, deserve more careful attention.

§(2): *The Three Kinds of Variational Principles*

Let us start with a short account on the framework in which the problem arises. Let M_4 be a four-dimensional C^∞ (Hausdorff, paracompact) differentiable manifold, endowed with a Lorentzian metric g and an affine torsionless connection Γ. Let ∇ be the covariant derivative with respect to Γ. Assume a Lagrangian density $\mathcal{L}_{MA}$ is given in M_4, depending on g, Γ, ∂g, $\partial\Gamma$, and, possibly, on a certain number of matter fields which we shall collectively denote by ϕ, together with their first derivatives $\partial\phi$. A metric-affine variational principle consists of the following prescription:

$$\delta_u\left(\int \mathcal{L}_{MA}(u, \partial u)\, d^4x\right) = 0 \tag{1}$$

where the variation δ_u is taken over the following set of independent variables:

$$u = (g, \Gamma, \phi) \tag{2}$$

The simplest relevant case is obtained by taking

$$\mathcal{L}_{MA}(u, \partial u) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma, \partial\Gamma)\,[-\det(g_{\mu\nu})]^{1/2} \tag{3}$$

where $R_{\alpha\beta}(\Gamma, \partial\Gamma)$ is the Ricci tensor of Γ. It is fairly well known[2] that the prescription (1) when applied to (3) gives, after some easy manipulations, the following Euler-Lagrange equations:

$$G_{\alpha\beta}(g, \Gamma, \partial\Gamma) = R_{(\alpha\beta)}(\Gamma, \partial\Gamma) - \tfrac{1}{2} g_{\alpha\beta}\,[g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial\Gamma)] = 0 \tag{4a}$$

$$\nabla_\sigma g_{\alpha\beta} = 0 \tag{4b}$$

[1] An English version of Palatini's paper was originally included herein as an Appendix. After this paper was submitted for publication, we found that an English translation by R. Hojman and C. Mukku had been already published in P. G. Bergmann and V. De Sabbata (eds.), *Cosmology and Gravitation*, Plenum Press, New York (1980), and so it has been omitted here.

[2] See, e.g., [1], Section 21.2, or [2], Chap. XII.

From (4b) it follows that Γ coincides with the Levi-Civita connection of g and, as a consequence, (4a) are the usual vacuum Einstein equations. The method of treating the metric g and the connection Γ as independent variables [yielding, of course, $10 + 40$ independent equations, like, e.g., Equations (4) in the vacuum case] is exactly what is usually cited as "Palatini's variational method".[3] However, as we shall clarify later, Palatini was far from such a formulation.

Although it is rather trivial, we stress that, in the metric-affine picture, no *a priori* relation can be imposed between g and Γ (and ϕ, too), since they have to be varied independently. In particular, one cannot take Γ to be the Levi-Civita connection of g, as it is sometimes erroneously assumed in literature. Actually, under this last assumption, one may rewrite $\mathcal{L}_{MA}(g, \Gamma, \partial\Gamma; \phi, \partial\phi)$ as a new density $\mathcal{L}_M(g, \partial g, \partial^2 g; \phi, \partial\phi)$ and the variational prescription (1) should be replaced by the following:

$$\delta_v\left(\int \mathcal{L}_M(v, \partial v, \partial^2 v)\, d^4x\right) = 0 \tag{5}$$

where the variation δ_v is taken over the set of independent variables

$$v = (g, \phi) \tag{6}$$

This prescription expresses what we should call a purely metric variational principle. As an example, let us examine the vacuum case described by the Lagrangian:

$$\mathcal{L}_M(v, \partial v, \partial^2 v) = R(g, \partial g, \partial^2 g)\, [-\det(g_{\alpha\beta})]^{1/2} \tag{7}$$

which is obtained from (3) by replacing Γ with the Levi-Civita connection γ of g. Then, the variational principle (5) yields exactly the ten vacuum Einstein equations:

$$G_{\alpha\beta}(g) = R_{\alpha\beta}(g) - \tfrac{1}{2}\, g_{\alpha\beta}\, [g^{\mu\nu}R_{\mu\nu}(g)] = 0 \tag{8}$$

[where $R_{\alpha\beta}(g)$ is the Ricci tensor of the Levi-Civita connection γ of g].

In the framework of purely metric variational principles, a slightly different viewpoint was assumed by Palatini in his original paper [4]. His ideas consist essentially in the following. Being $\gamma(g, \partial g)$ a known function of g and ∂g, one first calculates the variation $\delta\gamma$ as a function of the variation δg and observes that it is a tensor.[4] One obtains

$$\delta_v \gamma^{\alpha}_{\beta\mu}(g, \partial g) = \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial g_{\rho\sigma}}\, \delta g_{\rho\sigma} + \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial g_{\rho\sigma,\tau}}\, \delta g_{\rho\sigma,\tau} \tag{9}$$

[3] We list here some of the best-known references: [3, 1, 2].
[4] See Equation (8) of [4].

Using the well-known expression of the Riemann curvature tensor in terms of γ and $\partial\gamma$ one then expresses its variation as a function of $\delta\gamma$ (and, therefore, implicitly as a function of δg). From this one has the equation

$$\delta_v R_{\alpha\beta} = \delta\gamma^{\lambda}_{\alpha\beta;\lambda} - \delta\gamma^{\lambda}_{\alpha\lambda;\beta} \tag{10}$$

which expresses the variation $\delta_v R_{\alpha\beta}$. At this point one inserts $\Gamma = \gamma$ into (3) and instead of reexpressing explicitly $\mathcal{L}_{MA}$ as $\mathcal{L}_M$, one calculates implicitly the variation of $\mathcal{L}_M$ as follows:

$$\delta_v \mathcal{L}_M = \{G_{\alpha\beta}(g)\, \delta g^{\alpha\beta} + [\delta_v R_{\alpha\beta}(g)]\; g^{\alpha\beta}\}\; [-\det(g_{\mu\nu})]^{1/2} \tag{11}$$

To end up, one should finally insert the variation $\delta_v R_{\alpha\beta}(g)$ [given by (10)] into (11) and reexpress $\delta_v \mathcal{L}_M$ as a function of δg. However, the second term of $\delta_v \mathcal{L}_M$ is a pure divergence, which drops out by an integration by parts. Einstein equations (8) then follow as before. Remarkably enough, Equation (10) of Palatini's paper (as well as the method employed in [4] to obtain it) extends immediately to any affine torsionless connection.[5]

A third kind of variational procedure consists of the so-called purely affine variational principles. In these approaches one takes as independent variables the following quantities:

$$w = (\Gamma, \phi) \tag{12}$$

and gives accordingly a Lagrangian density $\mathcal{L}_A(\Gamma, \partial\Gamma; \phi, \partial\phi)$. The corresponding prescription is now

$$\delta_w\left(\int \mathcal{L}_A(w, \partial w)\, d^4x\right) = 0 \tag{13}$$

§ (3): *The Historical Perspective*

As early as 1914, well before the independent deduction by Einstein and Hilbert of the final equations of Einstein's gravitational theory,[6] it was clear to Einstein and Grossmann that variational principles should play an important role in setting up the theory (see [5]).

Lorentz [6], stimulated by a letter from Einstein dated August 16, 1914,[7] studied Hamilton's principle in general relativity both for particles and the electromagnetic field. He derived a formal covariant variational procedure for the gravitational field itself in which the Lagrangian density was assumed to depend on g and ∂g, in a way which was left completely unspecified.

[5]Provided one takes [4].8 as the direct definition of the variation.
[6]See [7], p. 25.
[7]See [7], p. 43.

The crucial period for the formulation of the theory was, as is well known, the fall of 1915. In his first note [8], presented on November 4, 1915, Einstein derived a preliminary incomplete set of equations, obtaining them from a purely metric variational principle. However, the treatment was not satisfactory, since the coordinate condition $[|\det(g)|]^{1/2} = 1$ was assumed and, moreover, the Lagrangian was not generally covariant. The final equations were later deduced by Einstein in his note [9], presented on November 25, in which no variational method was used. At about the same time, and independently of Einstein, Hilbert obtained Einstein's equations in an axiomatic framework based on the extensive use of a metric variational principle having (7) as Lagrangian density. His results were presented on November 20, in the famous note [10] to the Göttingen Academy. This date is therefore the birthday of the usual metric variational principle for general relativity. In [10] Hilbert took in consideration also the interaction with electromagnetism in the framework of Mie's theory. In early 1916, Einstein published his paper on the foundations of general relativity [11]. The derivation of the correct gravitational equations contained in this paper was still unsatisfactory, because Einstein used again a noncovariant variational principle.

The question was then reexamined by Lorentz in a remarkable series of papers [12] which appeared during the year 1916, shortly after the above memoirs.[8] In particular, in [12, III] Lorentz applied his general apparatus (developed in [6]) to the correct Hilbert Lagrangian (7), coupled with several kinds of matter terms having a greater generality than that considered by Hilbert in [10]. He also investigated the corresponding conservation laws. From Lorentz' paper one learns that also De Donder contributed to the matter.

It seems that these papers by Lorentz influenced Einstein more than that of Hilbert's,[9] probably because they were more general than Hilbert's as far as the matter content was involved. As a consequence, Einstein revisited the problem in his note [14] presented on November 26, 1916, with the title "Hamiltonsches prinzip und allgemeine Relativitäts Theorie." In this note he succeeded in giving up the restrictive coordinate conditions assumed earlier; moreover, he made "especially in contrast to Hilbert, as few restrictive assumptions as possible about the constitution of matter. . . ."[10] At this stage, we see that the purely metric variational approach to general relativity had reached a

[8] See [7], pp. 43 and 44.

[9] See [7], p. 44, Section 6.2.

[10] We have quoted here the translation given by Mehra in [7], end of p. 44. The original text reads as follows: ". . . Insbesondere sollen über die Konstitution der Materie möglichst wenig spezialisierende Annahmen gemacht werden, im Gegensatz besonders zur HILBERTschen Darstellung. . . ." It is interesting to compare the translation above and the original text with the following translation, by W. Perret and G. B. Jeffrey, given in [13], p. 167: ". . . we shall make as few specializing assumptions as possible, in marked contrast to Hilbert's treatment of the subject."

self-consistent, covariant, and satisfactory formulation, thanks to the efforts of Einstein, Hilbert, and Lorentz. Further improvements to this subject were due to Klein, who, at the very beginning of 1917, in the note [15] simplified Hilbert's calculations of the variational problem and dealt with the conservation laws. These were further reelaborated in [16] and [17] by relying on the then recent ideas of E. Noether.

Around the same years, a new line of mathematical thought began to develop. The first step was taken by Levi-Civita in his fundamental memoir [18],[11] where he introduced the notion of parallel transport along the curves of a given Riemannian manifold (isometrically embedded into a Euclidean space). This paper clarified the important geometrical meaning of the Christoffel symbols $\{^{\alpha}_{\beta\gamma}\}$, which are the coordinate representation of the geometrical object by now commonly known as Levi-Civita's connection. Levi-Civita's work, together with a paper by Hessenberg [19], were a source of inspiration for Weyl. In the first edition of his book [20], Weyl redefined the parallel transport of Levi-Civita in a truly intrinsic way, i.e., without resorting to embeddings. This was the basis for further extensions, undertaken in [21], and also discussed in the third edition of the book [22], 1929, in which the concept of parallel displacement was developed in manifolds without a metric structure. These were the (yet preliminary) foundations of the theory of (symmetric) linear connections. Around 1922, this theory was completed and generalized by Cartan, with the required formalizations and a further extension to contemplate nonsymmetric connections (see [23-26]).

§ (4): *An Historical Overview on the Metric-Affine and Affine V.P.*

We are finally in a position to discuss Palatini's paper in the historical context described in Section 2. The paper was contributed to the Circolo Matematico di Palermo on August 10, 1919, together with another paper about the foundations of the absolute differential calculus. In his paper [4], Palatini quotes the (purely metric) variational approach as developed by Hilbert in [10] and by Weyl in [21]. The aim of Palatini was to improve Hilbert's deduction of Einstein's field equations from variations with respect to g of the usual (metric) Lagrangian (7). The aim is to preserve the tensorial character of all the equations at each step of the deduction. There is a merit in this approach, since Palatini is able for the first time to show that the variations of Christoffel symbols constitute the coordinate components of a tensor, and moreover his method of varying the Riemann curvature tensor is independent of the particular choice of a symmetric connection. We should stress, however, that by no means did Palatini

[11] Completed during November 1916, but communicated on December 24 and published in the issues of 1917.

operate in a nonmetric framework, nor did he suggest in [4] a possible extension of his method to the case of an arbitrary symmetric affine connection. Nevertheless, one should notice that such an extension could have been done by Palatini himself, in view of the fact that he was acquainted with Weyl's pioneering ideas.

During the year 1920, Pauli, with the purpose of reviewing the state of the art of relativistic theories in his contribution [27] to the *Encyklopädie der mathematischen Wissenschaften*, collected most of the existing literature on the subject. In particular, Pauli refers to Palatini's paper in a footnote[12] which is of a technical nature, it being concerned with the expression of the variation δ_g of the gravitational action. In the same footnote, reference is made to p. 205 and 206 of the third edition [22] of Weyl's book, where Weyl presents an alternative derivation of the same equations based on Hilbert's and Einstein's ideas. The third edition [22] was completed during August 1919, i.e., almost at the same time of Palatini's paper. Therefore it is not surprising that in [22] there is no mention of [4]. Apparently, during 1920 there was an exchange of bibliography between Pauli and Weyl[13] and, as a consequence, in the fourth revised version [28] also Weyl refers to Palatini[14] by sketchilly recalling his approach.

One year later, stimulated by Weyl's ideas, people began to think about purely affine gravitational theories, i.e., theories based only on a given affine connection. Eddington was the first to move in this direction, with his 1921 paper [29]. In this paper, Eddington tried a better formalization and a generalization of Weyl's theory by starting from an affinely connected manifold. However, his approach should be still considered as a purely metric one because he introduces a metric through the knowledge of the Ricci tensor and varies with respect to it a suitably reformulated metric Lagrangian. A slightly different but essentially equivalent approach is followed also in his 1923 book [30].[15] This line of thought is the natural evolution of some preliminary ideas at the purely metric level, which Eddington worked out in 1920 and published in 1921 as a long mathematical supplement to the French edition [31] of his book *Space, Time and Gravitation*.

As we see, Eddington was very near to the formulation of a truly affine theory, but he missed his aim because he did not specify which equations govern the affine connection. As far as we know, this goal was shortly after reached by

[12] See [27], footnote 105, p. 621.

[13] We are not aware of any correspondence between the two of them which could support our opinion. Nevertheless, one feels such an impression by carefully comparing the bibliographies and the footnotes of [27], [22], and [28]. In fact, it is striking to observe that almost all the references added to Chap. IV in [28], with respect to the previous edition [22], also appear in [27]. As a last remark, we point out the following coincidence of dates: [27] was terminated in December 1920, [28] was terminated in November 1920.

[14] See [28], pp. 216 and 217.

[15] See [31], Sections 95–101. The relevant points are in Section 101, where the affine Lagrangian is rewritten under a metrical form.

Einstein. In a series of three papers published in 1923 (see [32–34]), he laid down in a very neat way the foundation of such a theory. He was the first who clearly pointed out that one should start from a Lagrangian depending only upon a connection, together with its first derivatives, and one should accordingly take variations with respect to the connection itself (see [32], p. 34).

At this point it is worthwhile to stop a moment to reflect about the physical motivations which stimulated the above developments. Weyl's theory, as well as Eddington's and Einstein's, were aimed at constructing field theories able to unify on a common geometrical ground both the gravitational and the electromagnetic phenomena. By this reason, "classical" general relativity had to be found as a particular limit of these new "unified" theories. As a result, also these unified theories should contain in an essential way a metric tensor $g_{\alpha\beta}$. In Weyl's theory the metric was given *a priori*. In Eddington's and Einstein's ones, the metric was obtained as a by-product, respectively, as the symmetric part of $R_{\alpha\beta}$ or as the symmetric part of $\partial\mathcal{L}/\partial R_{\alpha\beta}$. However, these scientists were not satisfied by the state of the art of unification in 1923. We can read this impression in a note added by Eddington to the 1924 reedition [35] of his book [30][16]: ". . . The theory is intensely formal as indeed all such action-theories must be, and I cannot avoid the suspicion that the mathematical elegance is obtained by a short cut which does not lead along the direct route of real physical progress. From a recent conversation with Einstein I learn he is of much the same opinion . . ."

Therefore, it is not surprising that Einstein himself considered again the whole problem in the paper [36] of 1925. Indeed, in the introduction to [36], one can read "Auch von meiner in diesen Sitzungsberichten (XVII, S.137, 1923) erschienenen Abhandlung, welche ganz auf EDDINGTONS Grundgedanken basiert war, bin ich der Ansicht, dass sie die wahre Lösung des Problems nicht gibt. Nach unablässigem Suchen in then letzten zwei Jahren glaube ich nun die wahre Lösung gerfunden zu haben . . ."[17] Unfortunately, also this attempt turned out to be unsatisfactory. On the other hand, this paper is crucial for our concern because it is exactly the first place in which the so-called Palatini method makes its appearance, already in a generalized form. In fact, in [36] Einstein assumes that a nonsymmetric connection $\Gamma^{\alpha}_{\mu\nu}$ and a generic contravariant 2-tensor density $g^{\mu\nu}$ are given in four-dimensional space. Then he states "Aus beiden bilden wir die skalare Dichte

$$\mathcal{H} = g^{\mu\nu} R_{\mu\nu}$$

[16] See [35], p. 257.

[17] ". . . I believe that also my preceding notes, published in this Sitzungsberichten (XVII, S.137, 1923), which were based on Eddington's ideas, do not contain the solution of the problem. After a very hard research during the last two years I now think I have got the correct solution . . ."

und postulieren, dass sämtliche Variationen des Integrals

$$\mathfrak{J} = \int \mathfrak{H}\, dx_1\, dx_2\, dx_3\, dx_4$$

nach den $g^{\mu\nu}$ und $\Gamma^{\alpha}_{\mu\nu}$ als unabhängigen (an den Grenzen variierten) Variablen verschwinden . . ."[18] The variation with respect to $g^{\mu\nu}$ gives the 16 equations $R_{\mu\nu} = 0$. The variation with respect to $\Gamma^{\alpha}_{\mu\nu}$ gives the other 64 equations. As a particular case, Einstein considers in some detail the case of symmetric $g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ (which he calls "pure gravitational field"). After some manipulations he proves that the 64 equations for $\Gamma^{\alpha}_{\mu\nu}$ reduce to the statement that $\Gamma^{\alpha}_{\mu\nu}$ is the Levi-Civita connection of the metric tensor associated with $g^{\mu\nu}$.[19] No mention of Palatini's paper is made.

It seems that Einstein at that time was not aware of the contents of Palatini's work, and moreover that the method of independently varying g and Γ was one of his own original ideas. We have no proof of this fact. However, our conjecture is strongly supported by considering the further developments of the "affair." In many of his later papers on relativity, Einstein refers explicitly to Palatini. However, not all the quotations are properly made and we found it interesting to follow their chronological "evolution." Seemingly, the first reference by Einstein to Palatini's method appears in the 1941 paper [37],[20] where the well-known relation (1.10) is quoted as due to Palatini. An analogous statement concerning a five-dimensional metric manifold appears in [39]. These quotations refer to the purely metric framework and, as such, they are perfectly correct. However, in 1946, Einstein and Straus incorrectly claim the following: ". . . For the variation according to the Γ we use the method which *has been established* by Palatini for the case of symmetric g and Γ . . ."[21] This statement concerns a variational metric–affine approach to unified field theories with nonsymmetric g and Γ. It is rather surprising that the method referred to by Einstein and Straus is a reformulation in more precise terms of Einstein's 1925 original approach. A few years later, in the Appendix II to the 1950 edition [41] of his book *The Meaning of Relativity*, Einstein refers to ". . . Palatini's device which can easily be extended to non symmetrical fields . . ."[22] A further and more precise reference to the fact that (1.10) may be generalized by allowing nonsymmetric fields appears in [42].

[18] "From these two quantities we build the scalar density $\mathfrak{H} = g^{\mu\nu}R_{\mu\nu}$. We postulate that any variation (vanishing at the boundary) of the integral $\mathfrak{J} = \int \mathfrak{H}\, dx_1\, dx_2\, dx_3\, dx_4$, taken with respect to $g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ as independent variables, vanishes . . ."

[19] See [36], pp. 416 and 417.

[20] For historical remarks on [37] see [38]. We point out that in [38] there is an incorrect reference to Palatini.

[21] See [40], p. 734.

[22] See [41], p. 140.

It is our opinion that in his late years Einstein was aware of his personal contribution to the method of varying g and Γ as independent variables. Nevertheless, he used to refer to Palatini without mentioning his own previous work, probably in order to follow the by then accepted custom. In fact, in most of the current literature on general relativity the use of quoting such a method as "Palatini's (variational) method" was growing up very rapidly in those years.[23] Since then, it seems that the custom has become deep rooted and many treatises on gravitation state more or less explicitly that the method of varying g and Γ independently was invented by Palatini in 1919. This error is unfortunately spreading out over the greater part of research papers about variational principles in relativity.

This happens in spite of some (marginal) precise reference to Einstein's priority, among which we might quote the following: H. Weyl, in the 1950 preface to the first American printing [45] of his book *Space, Time, Matter* (p. vi); J. L. Anderson, in a footnote on p. 345 of his book [46], and Pauli in a note added to the 1958 edition of his book [47]. We hope that the present paper will also contribute to avoid any further misunderstanding.

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